A STUDY OF FUNCTIONAL DIFFERENTIAL AND INTEGRO-DIFFERENTIAL EQUATIONS CONTAINING MEASURES

BY

RAI RAMJEE SHARMA

POST GRADUATE OFFICE
This thesis has been approved
for the award of the Degree of
Doctor of Philosophy (Ph.D.)
in accordance with the
regulations of the Indian
Institute of Technology Kanpur
Dated: 29/1/71

DEPARTMENT OF MATHEMATICS

Indian Institute Of Technology, Kanpur

JULY, 1970.

MATH-1970-D-SHA-STU

JUNE '76

16 JUN 1972

I.I.T. KANPUR
CENTRAL LIBRARY
Acc. No. A 19835

A STUDY OF FUNCTIONAL DIFFERENTIAL AND INTEGRO - DIFFERENTIAL EQUATIONS CONTAINING MEASURES

A THESIS

Submitted in Partial Fulfilment of the Requirements for

THE DEGREE OF DOCTOR OF PHILOSOPHY

by

Rai Ramjee Sharma

to the

Department of Mathematics

INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

JULY, 1970.

TO THE MEMORY OF MY FATHER

CERTIFICATE

Certified that the thesis entitled 'A Study of Functional Differential and Integro-Differential Equations Containing Measures' by Mr. R. R. Sharma has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

P.C. Das

Dr. P. C. Das
Department of Mathematics
Indian Institute of Technology,
Kanpur.

July 12, 1970.

ACKNOWLEDGMENTS

I am deeply indebted to Dr. P. C. Das for introducing to me Functional Differential Equations and for supervising this work. I wish to thank my friends Mr. G. D. Lakhani, Mr. S. K. Bhatt and Mr. N. Parhi for many fruitful conversations on various topics. A special word of thanks is due to my wife whose cooperation was a great help in completion of this work. Finally, I thank Mr. S. L. Jha for typing this thesis with interest.

R R Sharma

July, 1970. (R. R. Sharma)

C O N T E N T S

LIST OF SYMBOLS

$f\ (t-)$	Left-hand limit of f at t
$f\ (t+)$	Right-hand limit of f at t
$\not\to$	Does not converge to
$\in$	Belongs to
$\notin$	Does not belong to
inf	Infimum
sup	Supremum
$\overline{\lim}$	Upper limit
$\underline{\lim}$	Lower limit
$x^{(k)}(.)$	Sequence of functions
μ	Measure
LS	Lebesgue - Stieltjes
df	LS-measure induced by f
$\lvert\mu\rvert$	Total variation of μ
$v(f,I)$	Total variation of f on the interval I
v_f	Total variation function of f
$\emptyset$	Empty set
$A \subset B$	A is a subset of B
$\{ x \mid P \}$	Set of all elements x which have the property P
$A \cup B$	Union of sets A and B
$A \cap B$	Intersection of sets A and B
$A - B$	Difference of sets A and B

$-A$	Complement of set A
$\bigcup_{\alpha\in\Gamma} A_\alpha$, $\bigcap_{\alpha\in\Gamma} A_\alpha$	Union, intersection of class of sets $\{A_\alpha \mid \alpha\in\Gamma\}$, Γ an index set
$\bigcup_{i=1}^{\infty} A_i$, $\bigcap_{i=1}^{\infty} A_i$	Union, intersection of sequence of sets $\{A_i \mid i=1,2,\ldots\}$
$f(A)$	$\{f(x) \mid x\in A\}$
$\int_E f\, d\mu$	Integral over E of f with respect to measure μ
$\int_a^b f(t)dg(t)$	Integral $\int_I f d\mu$ where $I = (a,b)$ and μ is LS-measure induced by g
a.e.	Almost everywhere
iff	If and only if

TABLE OF SPACES

SYMBOL	THE SET OF	A TYPICAL MEMBER	NORM
R	real numbers	t	
R^n	n-tuples of real numbers	$\xi=(\xi^1,\ldots,\xi^n)$	$\lvert\xi\rvert=\Sigma_{i=1}^{n}\lvert\xi^i\rvert$
$\mathfrak{M}$	n x m matrices of real numbers	$M=(M^i_j)$	$\lvert M\rvert=\Sigma_{i=1}^{n}\Sigma_{j=1}^{m}\lvert M^i_j\rvert$
$C_c(X)$	continuous complex functions on X whose support is compact, X a topological space	ψ	$\lvert\psi\rvert=\sup_{x\in X}\lvert\psi(x)\rvert$
$C_c^\infty(\Omega)$	complex functions on Ω with infinitely many partial derivatives and having compact support, Ω an open set of R^n	ψ	$\lvert\psi\rvert=\sup_{\xi\in\Omega}\lvert\psi(\xi)\rvert$
$BV(I)$	scalar functions of bounded variation on I, I an interval of R	φ	$\lvert\varphi\rvert_I=v(\varphi,I)+\lvert\varphi(a+)\rvert$ where a is left end point of I
$NBV(I)$	right continuous functions $\varphi\in BV(I)$ with $\varphi(a+)=0$, where a is left end point of I	φ	$\lvert\varphi\rvert_I=v(\varphi,I)$
$BV(I)_n$	vector functions with values in R^n whose individual components $\in BV(I)$	$\varphi=(\varphi^1,\ldots,\varphi^n)$	$\lVert\varphi\rVert_I=\Sigma_{i=1}^{n}\lvert\varphi^i\rvert_I$

SYNOPSIS

'A Study of Functional Differential and Integro-Differential Equations Containing Measures', a thesis submitted in partial fulfilment of the requirements for the Ph.D. degree by Rai Ramjee Sharma to the Department of Mathematics, Indian Institute of Technology, Kanpur.

When a system described by ordinary differential equation $dx/dt = f(t,x)$ is acted upon by perturbation, the perturbed system is generally given by ordinary differential equation of the form $dx/dt = f(t,x) + G(t,x)$ where the perturbation term $G(t,x)$ is assumed to be continuous or integrable and as such the state of the system changes continuously with respect to time. But in physical systems one cannot expect the perturbations to be well-behaved and it is therefore important to consider the case when the perturbations are impulsive. This will give rise to equations of the form

$$Dx = f(t,x) + G(t,x)\ Du \tag{1}$$

where Du denotes the distributional derivative of function $u(t)$. If u is a function of bounded variation, Du can be identified with a Stieltjes measure and will have the effect of suddenly changing the state of the system at the points of discontinuity of u. By a solution $x(.)$ of (1) is meant a function of bounded variation whose distributional derivative Dx satisfies the equation (1). W.W.Schmaedeke in his paper 'Optimal Control Theory for Nonlinear Vector Differential

Equations containing Measures' (SIAM J. on Control, Vol.3 (1965), pp. 231-280) considered the theory of equation (1) in the special case when G depends only on t, i.e. $G(t,x)=G(t)$ and is assumed to be a continuous function of t in order to apply the methods of Riemann-Stieltjes integrals in the subsequent analysis. The present thesis deals with the existence and some other properties of the solutions of equation (1) and its generalisations to the following functional differential and integro-differential equations:

$$Dx = f(t,x_t) + G(t,x_t)\ Du \tag{2}$$

$$Dx = f(t,x_t) + G(t,x_t)\ Du + \int_{t_0}^{t} H(x(t+t_0-s))\, d\lambda(s) \tag{3}$$

where x_t represents the restriction of the function $x(s)$ on the interval $p(t) \leq s \leq q(t)$, p and q being functions with the property $p(t) \leq q(t) \leq t$ for each $t \geq t_0$. In the analysis that follows, the possibility that $G(t,x_t)$ and $u(t)$ may have common discontinuities, unlike the case in Schmaedeke's work, renders the method of Riemann Stieltjes integrals unapplicable and Lebesgue Stieltjes integrals are therefore used.

In Chapter 2 the equation (3) is converted to an equivalent integral equation by a careful use of the integration by parts procedures for L.S. integrals which are not valid in general and through this various theorems concerning existence and uniqueness of solutions of (3) are established.

Chapter 3 deals with a special case of (3) where G is independent of x, i.e. $G(t,x_t) = G(t)$. A constructive proof of the local existence and uniqueness theorem is given. It also treats the continuous dependence of solution on the initial function and the right hand side of the equation.

In Chapter 4 equation (1) is considered. A local existence theorem is proved by relaxing the hypotheses on G in the corresponding theorem for (3). The problem of stability for the solution of (1) is also considered.

In Chapter 5 an optimal control problem with a vector-valued cost functional $J(u) = (J^1(u), J^2(u), \ldots, J^{\ell}(u))$ is formulated for a control process governed by (2). A partial ordering is defined in the 'cost' of the controls as follows : $J(u^*) \leq J(u)$ if $J^i(u^*) \leq J^i(u)$, $i = 1,2,\ldots \ell$. A control $u^*(t)$ is defined to be optimal if $J(u) \leq J(u^*)$ implies $J(u) = J(u^*)$. The existence of an optimal control is established under suitable conditions.

CHAPTER 0

INTRODUCTION

0.1 FUNCTIONAL DIFFERENTIAL AND INTEGRO-DIFFERENTIAL EQUATIONS

Some important problems of physics and technology, where hereditary effects play an important role, led to the investigation of differential equations with deviating argument and integro-differential equations ([2] , [37]) . In the last two decades such equations have received abundance of applications not only in the various branches of physics and technology but also in economics and biological sciences (see the bibliographies in [3] and [27]). This has caused much interest in the theory of differential equations with ~~deviating arguments~~ and of integro-differential equations. In the last 15-20 years an enormous number of papers have been devoted to such equations - particularly in USSR (see [36] and the bibliographies in [12] , [27]). In particular, many results from the theory of ordinary differential equations were translated to differential equations with deviating arguments and to more general functional differential equations, and integro-differential equations which allowed to construct general theory of such equations.

'Differential equations with deviating arguments' are differential equations in which the unknown function and its derivatives appear with various values of the argument. For example :

$$\dot{x}(t) = f(t,x(t),x(t-h)) \tag{0.1.1}$$

$$\dot{x}(t) = f(t,x(t),x(t-h_1(t)),x(t-h_2(t)), \dots, x(t-h_m(t))) \tag{0.1.2}$$

$$\ddot{x}(t) = f(t,x(t),\dot{x}(t),x(t-h(t)), \dot{x}(t-h(t))) \tag{0.1.3}$$

$$\dot{x}(t) = f(t,x(t),x(t-h_1),\dot{x}(t-h_2)) \tag{0.1.4}$$

$$\ddot{x}(t) = f(t,x(t),\dot{x}(t),x(t-h(t)) \dot{x}(t-h(t)), \ddot{x}(t-h(t))) \tag{0.1.5}$$

If in a differential equation with deviating argument the highest-order derivative of the unknown function appears for just one value of the argument then the equation is said to be of 'lag type' or 'advanced type' according as this argument is not less than or not greater than all arguments of the unknown function and its derivatives appearing in the equation. For example, equations (0.1.1), (0.1.2) and (0.1.3) are equations (i) of lag type if $h>0$ in (0.1.1), $h_j(t) \geq 0$ for $j=1,2,\dots, m$ in (0.1.2) and $h(t) \geq 0$ in (0.1.3) ; and (ii) of advanced type if $h < 0$ in (0.1.1), $h_j(t) \leq 0$ for $j=1,2,\dots, m$ in (0.1.2) and $h(t) \leq 0$ in (0.1.3). All other differential equations with deviating arguments, for examples equations (0.1.4) and (0.1.5), are called equations of the neutral type. An analogous classification applies also for systems of equations.

Consider now the following integro-differential equation with deviating argument :

$$\overset{(n)}{x}(t) = f(t,x(t),\dot{x}(t),\ldots,\overset{(n-1)}{x}(t),$$
$$x(t-h(t)),\ldots,\overset{(n-1)}{x}(t-h(t)))$$
$$+\int_a^{\sigma} g(t,s,x(s),\dot{x}(s)\ldots,\overset{(k)}{x}(s),$$
$$x(s-h(s)),\dot{x}(s-h(s)),\ldots,$$
$$\overset{(\ell)}{x}(s-h(s))ds \qquad (0.1.6)$$

$$\left(\overset{(n)}{x} \equiv \frac{d^n x}{dt^n}\right)$$

where $\sigma = t$, or $\sigma = b = \text{const}$.

If $\max\{k,\ell\} < n$, then the classification of integro-differential equation (0.1.6) is possible to do according to the differential part, as it was done above for differential equations with deviating argument. If $\max\{k,\ell\} \geq n$ then for the classification we have to take into account the integral part. In this case the equation cannot be classified by the external appearance. Consider, for example, the following equation

$$\dot{x}(t)=x(t) + \int_0^1 B(s)\ddot{x}(t-s)ds$$

where the function B is twice differentiable. Integrating $\int_0^1 B(s)\ddot{x}(t-s)ds$ twice by parts, we obtain the integro-differential equation

$$\dot{x}(t)=x(t)+B(1)\dot{x}(t-1)-B(0)\dot{x}(t)$$
$$-\dot{B}(1)x(t-1)+\dot{B}(0)x(t)+\int_0^1 \ddot{B}(s)x(t-s)ds.$$

From this it is seen that we can obtain different types of equations with different $B(1)$, $B(0)$, $\acute{B}(1)$, $\acute{B}(0)$. For the classification in such cases it is necessary to start from the solutions of integro-differential equations. If the solutions get smoothened then the equation is said to be of lag type. If the smoothness deteriorates then it is of advanced type.

In this monograph we shall be concerned with equations of lag type which are most often encountered in applications. Since t usually represents the time in applications, we shall be interested in continuing a solution in the direction of increasing t. Equation (0.1.1) with $h>0$ represents the simplest type of equation with time lag. In (0.1.2), where $h_j(t)\geq 0$, the values of the function x appear with different time lags and these time lags depend on t. For a more general equation with time lag, we may consider

$$\dot{x}(t) = f(t, x(t+\theta), -h\leq\theta\leq 0) \qquad (0.1.7)$$

where for each fixed t, f is a functional defined on a function space given in $[-h,0]$, $h>0$. The function f therefore depends on the whole behaviour of function x on $[t-h,t]$. Equations of the form (0.1.7) are called functional differential equations. Equation (0.1.6) can also be generalized to a functional integro-differential equation if $t-h(t)$ is replaced by $t+\theta, -h\leq\theta\leq 0$.

We shall now formulate the initial value problem for the system of functional integro-differential equations

$$\frac{dx^i}{dt} = f^i(t,x^1(t+\theta),x^2(t+\theta),\ldots,x^n(t+\theta),-h\leq\theta\leq 0)$$
$$+\int_a^{\sigma} g^i(t,s,x^1(s),\ldots,x^n(s))ds,$$
$$i=1,2,\ldots,n$$

or in vector form

$$\frac{dx}{dt} = f(t,x(t+\theta),-h\leq\theta\leq 0)+\int_a^{\sigma} g(t,s,x(s))ds \qquad (0.1.8)$$

where $\sigma=b$, or $\sigma=t$. Let a n-vector function $\varphi=(\varphi^1,\cdots,\varphi^n)$ defined on $[t_0-h,t_0]$ be given. The initial value problem consists of determining an absolutely continuous n-vector function $x(.)$ which satisfies (o.1.8) a.e. for $t>t_0$ and is such that

$$x(t)=\varphi(t) \text{ for } t_0-h\leq t\leq t_0 .$$

The function φ is called the 'initial function'.

The initial value problem is formulated above with any t_0. However, for integro-differential equations it is crucial to know the position of t_0, more particularly whether $t_0=a$ or $t_0\neq a$ where a is the left limit of integration. If $t_0>a$, the equation (0.1.8) can be reduced to one with $t_0=a$; if $t_0<a$, equation (0.1.8) is of advance type which may not have a solution. That is why existence theorems for functional integro-differential equations where $t_0\neq a$ to our knowledge is absent in literature. The initial value problem for $t_0=a$ appears quite often in the application where heredity into account and is

solved rather easily. Existence theorems in these cases are obtained by bringing the equations to integral form and then using the usual methods of integral equations such as contraction mapping principle, Schauder's fixed point theorem and methods of successive approximation.

Differential equations with deviating arguments are extensively investigated by Bellman and Cooke [2], El'sgol'ts [8,9], Myshkis [24], and Pinney [30]. Functional differential equation of the form (0.1.7) is investigated by Halanay [13] and Krasovskiĭ [18]. The equation (0.1.7) with $[-h,0]$ replaced by $(-\infty,0]$ has been studied by Oguztoreli [27]. For contributions in integro-differential equations with deviating argument one can refer to [36]. A survey of methods, techniques and related questions pertaining to such equations is presented by Gromova [12].

0.2 OPTIMAL CONTROL PROBLEM

In the theory of optimal control we quite often consider the following problem. Let there be given :

(i) a set $Q \subset R^m$,

(ii) a function f from $R \times R^n \times Q$ to R^n,

(iii) a function f^o from $R \times R^n \times Q$ to R,

(iv) an interval $[t_1, t_2] \subset R$,

(v) a family $\mathcal{J}$ of closed sets $T_t \subset R^n$ defined for $t \in [t_1, t_2]$,

(vi) a real number $t_o \leq t_1$,

(vii) a point $x_0 \in R^n$

(viii) a set U of functions u defined on $t_0 \leq t \leq \bar{t}$ where $\bar{t} \in [t_1, t_2]$ with values in Q.

Let U_0 be the subset of U consisting of such functions for which the corresponding solution x of

$$\frac{dx}{dt} = f(t,x(t),u(t)), \quad x(t_0)=x_0 \qquad (0.2.1)$$

exists for $t_0 \leq t \leq \bar{t}$ and satisfies $x(\bar{t}) \in T_{\bar{t}}$. The optimal control problem consists in finding u in U_0 for which the functional

$$J(u) = \int_{t_0}^{\bar{t}} f^0(t,x(t),u(t))dt \qquad (0.2.2)$$

takes its smallest value. A particular case of optimal control problem is 'time-optimal problem' in which $f^0 \equiv 1$ (so that $J(u)=\bar{t}-t_0$=transfer time) and $t_0=t_1$.

The family $\mathcal{J}$ is called the target set, the functions u of U are called (admissible) control functions, Q is the control set (or control region), the solution x of (0.2.1) is the trajectory corresponding to u, and J is called the cost functional. A pair u, x which minimizes (0.2.2) is called an optimal solution, u is an optimal control , and x is an optimal trajectory.

Generally the sets T_t vary continuously with t. The sets T_t may be independent of t, or each one may consist of a single point. In the first case we have a fixed target, in the second case the target is a single point $\xi(t)$ in R^n. Other problems are obtained when

(i) the initial point x_o is free to move on a given manifold in R^n; (ii) t_o is free ; (iii) x is restricted to have values in a region $X \neq R^n$. The set Q (and also X) need not be open. In fact, many technical problems require the control set Q to be compact. In such a case the optimal control may occur on the boundary of Q. Thus the optimal control problems fall outside the realm of the classical calculus of variations (see [31], Chapter V)

For the principal results dealing with the existence of optimal control one can refer to [20] and [35] and the references given therein. Some of these results have been extended by Oguztöreli [27] to control systems characterized by

$$\begin{aligned} \frac{dx}{dt} &= f(t,u(t),x(t+\theta), \alpha - t \leq \theta \leq 0) \\ x(t) &= \varphi(t) \text{ for } t \in [\alpha, t_o] \end{aligned} \qquad (0.2.3)$$

For formulation of optimal control problem in this case one is required to replace (0.2.1) by (0.2.3), and x_o by $\varphi(t)$ in the problem formulated above.

In an optimal process where optimal control with respect to a given cost functional $J_1(u)$ is not unique, one can introduce a second cost functional $J_2(u)$ and choose a control which minimizes $J_2(u)$ among the controls which minimize $J_1(u)$. If optimal control obtained this way is not unique, the above process

can be continued by introducing more cost functionals till the unique optimal control is obtained. A hierarchical optimal problem of this kind is considered by Chyung[4] .

0.3 SYSTEMS CONTAINING MEASURES

In the systems given by ordinary differential equations the values of the state variables change continuously with respect to time. When a physical system contain impulses, there occur discontinuous changes in the state variables of the system. Examples of such systems are pulse frequency modulation systems and models for biological neural nets. (See Pavlidis [28,29] and the references given therein). The ordinary differential equations are not suitable to deal with the systems containing impulses. Such systems, in fact, give rise to equations of the form

$$Dx = f(t,x) + G(t,x)Du \tag{0.3.1}$$

where Du denotes the distributional derivative of the function u. If u is a function of bounded variation, Du can be identified with a Stieltjes measure and has the effect of suddenly changing the state of the system at the points where u is discontinuous. For example if u is the Heaviside function

$$u(t) = \begin{cases} 0 \text{ for } t<0 \\ 1 \text{ for } t\geq 0 \end{cases}$$

then Du is the Dirac measure (known as Dirac δ-function by physicists and technologists). Schmaedeke [34] has

considered the theory of equation (0.3.1) in the special case when G is independent of x and is a continuous function of t only. He also considered the existence of optimal control for a control system given by

$$Dx = f(t,x,u) + G(t)\ Du \tag{0.3.2}$$

But the equation (0.3.1) has a more natural interpretation as perturbation of the ordinary differential equation $dx/dt=f(t,x)$ where $G(t,x)Du$ represents an impulsive perturbation. Usually we are concerned with stability where the perturbations are continuous or are sufficiently smooth so that the solutions are continuous. But in nature all perturbations cannot be expected to behave so well. So the perturbations of the type $G(t,x)Du$ are more realistic. Barbashin [1] and Zabalishchin [40] have initiated consideration of stability with respect to impulsive perturbations.

0.4 OUTLINE OF THE THESIS

Chapters 2,3,4,5 form the main body of the present thesis, and Chapter 1 contains the requisite mathematical equipment which is not a part of any theory of ordinary differential equations.

Chapter 2 deals with the existence and uniqueness of solutions of the system of functional integro-differential equations

$$Dx^i(t)=f^i(t,x_t^1,\ldots,x_t^n)+\sum_{j=1}^{m} g_j^i(t,x_t^1,\ldots x_t^n)Du^j(t)$$

$$+\int_{t_o}^{t}\sum_{j=1}^{m} h_j^i(x^1(t+t_o-\tau),\ldots,$$

$$x^n(t+t_o-\tau))d\lambda^j(\tau) \quad \text{for } t>t_o,$$

$$(0.4.1)$$

$$x^i(t)=\varphi^i(t) \quad \text{for } \alpha\leq t\leq t_o$$

$$(i=1,2,\ldots,n)$$

where x_t^i represents the restriction of the function $x^i(s)$ on the interval $p(t)\leq s\leq q(t)$, p and q being real functions with the property $\alpha\leq p(t)\leq q(t)\leq t$ for each $t\geq t_o$; for each fixed t, f^i and g_j^i are functionals defined on the space BV($[p(t),q(t)]$) ; $u^j(t)$ and $\lambda^j(t)$ are functions of bounded variation and are also right continuous ; and the derivatives $Dx^i(t)$ and $Du^j(t)$ are the distributional derivatives of $x^i(t)$ and $u^j(t)$ respectively. For brevity we write (0.4.1) in the vector form

$$Dx(t)=f(t,x_t)+G(t,x_t)Du(t)$$
$$+\int_{t_o}^{t} H(x(t+t_o-\tau))d\lambda(\tau) \quad \text{for } t>t_o \qquad (0.4.2)$$
$$x(t)=\varphi(t) \quad \text{for } \alpha\leq t\leq t_o$$

where $G(t,x_t)$ and $H\,(x(t+t_o-\tau))$ are n x m matrices. With different choice of functions p and q, equation (0.4.2) can be transformed into various integro-differential equations of lag type.

The consideration of equation (0.4.2) is motivated by [34] and [21]. In the optimal control problem considered in [34] the coefficient G of the impulsive controllers (see equation (0.3.2)) may be expected in many cases to depend not only on time t but also on the state variables x. Moreover the system may have hereditary effect. This consideration necessitates the study of the control system described by

$$Dx(t) = f(t,x_t,u(t))+G(t,x_t,u(t))Du(t) \qquad (0.4.3)$$

If G depends only on t, and $p(t)=q(t)=t$ so that $x_t=x(t)$, then (0.4.3) coincides with (0.3.2). Next, if in (0.4.2) f depends only on t, and $G=0$ then we obtain

$$Dx(t)=f(t)+\int_{t_o}^{t} H(x(t+t_o-\tau))d\lambda(\tau) \qquad (0.4.4)$$

In this case the solution can be taken to be an absolutely continuous function and then the distributional derivative Dx coincides with ordinary derivative dx/dt. Such an equation occurs in reactor dynamics and has been considered in [21].

Coming back to (0.4.2), we shall first define what is meant by a solution of it. Let S be a domain (an open connected set) in R^n and let $BV(I,S)$ denote the set of all vector-functions with values in S whose individual components belong to $BV(I)$. By a solution of (0.4.2) on $I = [\alpha,\sigma]$ or $[\alpha,\sigma)$ with the initial function $\varphi \in BV([\alpha,t_o],S)$ we mean a function $x \in BV(I,S)$

which is right continuous on (t_o, σ), satisfies the second equation of (0.4.2) on $[\alpha, t_o]$, and whose distributional derivative on (t_o, T) satisfies the first equation of (0.4.2) for any $T > t_o$ in I. To obtain the existence theorems, (0.4.2) has first been transformed into an integral form. More precisely, it has been proved in Theorem 1 that x is a solution of (0.4.2) iff it is a solution of the integral equation

$$x(t) = \begin{cases} \varphi(t) \text{ for } \alpha \leq t \leq t_o \\ \varphi(t_o) + \int_{t_o}^{t} f(s, x_s)\,ds + \int_{t_o}^{t} G(s, x_s)\,du(s) \\ \qquad + \int_{t_o}^{t} \left\{ \int_{t_o}^{s} H(x(s+t_o-\tau))\,d\lambda(\tau) \right. ds \\ \qquad\qquad \text{for } t > t_o \end{cases} \tag{0.4.5}$$

where it is assumed that for each x, $f(t, x_t)$ is Lebesgue measurable function of t and $G(t, x_t)$ measurable with respect to the Lebesgue-Stieltjes measure generated by u. The integrals are taken in the sense of Lebesgue and Lebesgue-Stieltjes integrals. This requires in the proof of Theorem 1 a careful use of integration by parts formula since it is not valid in general for Lebesgue-Stieltjes integrals. In [34], since the function G(t) is assumed to be continuous, the methods of Riemann-Stieltjes integrals were sufficient. But in the present case if $G(t, x_t)$ and u(t) have common points of discontinuity then $\int G(s, x_s)du(s)$ is not defined as R.S. integral and thus the methods of R.S. integrals are not applicable here. In Theorem 2 ,

local existence and uniqueness of solution of (0.4.2) has been proved, by the application of principle of contraction mapping, assuming that H is locally Lipschitzian on S, f and G are locally Lipschitzian in x, and some other conditions. Using a method of successive approximation, Theorem 3 shows that Lipschit conditions are not necessary for the local existence of solution. Theorem 4 is an extended existence theorem.

In Chapter 3, a special case of (0.4.2) when G is independent of x, i.e. $G(t,x_t) = G(t)$, is considered. Theorem 5 gives a constructive proof of the local existence and uniqueness of solution (under conditions similar to those in Theorem 2) using Picard's successive approximation procedure. Theorems 6 and 7 deal with the continuous dependence of solution on the initial function and the right hand side of the equation.

Chapter 4 deals with the equation (0.3.1). It is shown in Theorem 8 that in this case it is possible to prove the existence theorem by relaxing the hypotheses on G. Regarding (0.3.1) as perturbation of ordinary differential equation $dx/dt = f(t,x)$, Theorem 9 treats the stability of (0.3.1) when the stability of unperturbed system is known.

Lastly in Chapter 5, an optimal control problem with a vector valued cost functional $J(u)=(J^1(u),\ldots\ldots, J^{\ell}(u))$ is formulated for the process governed by (0.4.3). A partial ordering is defined in the 'costs' of the controls as follows :

$$J(u^*)\leq J(u)$$

if $\quad J^k(u^*)\leq J^k(u), \quad k = 1,2,\ldots\ldots,\ell$.

A control $u^*(t)$ is defined to be optimal if

$$J(u)\leq J(u^*) \text{ implies } J(u) = J(u^*).$$

The existence of an optimal control is treated in Theorem 10.

CHAPTER 1

PRELIMINARIES

It is the purpose of this chapter to recall certain notions and results of analysis which are used in the subsequent chapters.

1.1 FUNCTIONS OF BOUNDED VARIATION

A real variable will be denoted by t. Let I be an interval, finite or infinite, of the real line. If φ is a scalar (real or complex) function on the interval I, its total variation function v_φ is defined by

$$v_\varphi(t)=\sup \sum_{i=1}^{N} |\varphi(t_i)-\varphi(t_{i-1})|, t \in I$$

where the supremum is taken over all N and over all choices of $\{t_i\}$ such that $t_i \in I$ and

$$t_o<t_1< \dots <t_N=t$$

In general,

$$0 \leq v_\varphi(t) \leq v_\varphi(t') \leq \infty , \quad (t<t').$$

This implies that if v_φ is a bounded function, then

$$v(\varphi, I)=\lim_{t \to b-} v_\varphi(t)$$

exists and is finite, where b is the right end point of I. In this case φ is said to be of bounded variation on I, and v (φ, I) is called the total variation of φ on I. If φ is a function of bounded variation on I then the left-hand limit $\varphi(t-)$ exists for every t except the left end point

of I, the right-hand limit $\varphi(t+)$ exists for every t except the right end point of I, and the set of points at which φ is discontinuous is atmost countable. Also, if $t<t'$ then

$$|\varphi(t')-\varphi(t)| \leq v_\varphi(t')-v_\varphi(t)$$

The class of all scalar functions defined and of bounded variation on I is denoted by BV(I). If $\varphi \in BV(I)$ the norm of φ is defined by

$$|\varphi|_I = v(\varphi, I) + |\varphi(a+)| \qquad (1.1.1)$$

where a is the left end point of I. With this norm BV(I) is a Banach space. A function $\varphi \in BV(I)$ is called normalized if φ is right-continuous at each interior point of I and $\varphi(a+)=0$ where a is the left end point of I. The class of all such functions is denoted by NBV(I) and is also a Banach space with the norm given by

$$|\varphi|_I = v(\varphi, I) \qquad (1.1.2)$$

If $\varphi \in NBV(I)$, then $v_\varphi \in NBV(I)$. If $\varphi \in BV(I)$, then there is a unique constant c and a unique function $\psi \in NBV(I)$ such that

$$\varphi(t) = c+\psi(t)$$

at all points of continuity of φ. Also, $v(\psi, I) \leq v(\varphi, I)$.

According to a theorem of Helley, if an infinite family $\mathfrak{F}$ of functions $\in BV([a,b])$ is such that all functions of the family and total variation of all functions of the family are uniformly bounded

then there exists a sequence $\{\varphi_k\}$ in the family $\mathcal{F}$ which converges at every point of $[a,b]$ to some function $\varphi \in BV([a,b])$; moreover,

$$v(\varphi, [a,b]) \leq \underline{\lim}_{k\to\infty} v(\varphi_k, [a,b]) \qquad (1.1.3)$$

A function φ on I is said to be absolutely continuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\sum_{i=1}^{N} |b_i - a_i| < \delta \text{ implies } \sum_{i=1}^{N} |\varphi(b_i) - \varphi(a_i)| < \varepsilon$$

whenever (a_i, b_i), $i=1,2,\ldots, N$ are disjoint subintervals of I. If φ is absolutely continuous function on a bounded interval I then $\varphi \in BV(I)$.

The n-dimensional Euclidean space will be denoted by R^n and the norm of a vector $\xi = (\xi^1, \ldots, \xi^n) \in R^n$ will be defined by

$$|\xi| = \sum_{i=1}^{n} |\xi^i| \qquad (1.1.4)$$

Let $\mathcal{M}$ denote the set of n x m matrices of real numbers. The norm of a matrix $M = (M_j^i) \in \mathcal{M}$ will be defined by

$$|M| = \sum_{i=1}^{n} \sum_{j=1}^{m} |M_j^i| \qquad (1.1.5)$$

By $BV(I)_n$ will be denoted the space of all vector functions φ defined on I with values in R^n whose individual components $\in BV(I)$. The norm of φ is

$$\|\varphi\|_I = \sum_{i=1}^{n} |\varphi^i|_I = \sum_{i=1}^{n} \{v(\varphi^i, I) + |\varphi^i(a+)|\}$$
$$= v(\varphi, I) + \varphi(a+) \tag{1.1.6}$$

where a is the left end point of I, and $v(\varphi, I)$ is the total variation of φ on I defined by

$$v(\varphi, I) = \sum_{i=1}^{n} v(\varphi^i, I) \tag{1.1.7}$$

With this norm $BV(I)_n$ is a Banach space. The space $NBV(I)_n$ consists of those functions $\in BV(I)_n$ whose individual components $\in NBV(I)$. $NBV(I)_n$ is a Banach space with norm given by

$$\|\varphi\|_I = \sum_{i=1}^{n} v(\varphi^i, I) = v(\varphi, I). \tag{1.1.8}$$

1.2 COMPLEX MEASURES

We shall skip over the well known concept of 'positive measure' (usually called 'measure') and integration with respect to this measure, and shall recall some definitions and results concerning complex measures only.

Let $\mathcal{E}$ be a σ-algebra in a set X, i.e. a collection of subsets of X which contains the empty (void) set, the complement (relative to X) of each of its members and a countable union of its members. A countable collection $\{E_i\}$ of members of $\mathcal{E}$ is called a partition of E if $E_i \cap E_j = \emptyset$ whenever $i \neq j$, and if $E = \bigcup_{i=1}^{\infty} E_i$. A complex measure μ on $\mathcal{E}$ is a complex function

on $\mathcal{E}$ such that

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i) \qquad (E \in \mathcal{E})$$

for every partition $\{E_i\}$ of E. We can similarly define real measures which, infact, form a subclass of the complex ones.

The total variation $|\mu|$ of a complex measure μ on $\mathcal{E}$ is a set function on $\mathcal{E}$ defined by

$$|\mu|(E) = \operatorname{Sup} \sum_{i=1}^{\infty} |\mu(E_i)| \qquad (E \in \mathcal{E}),$$

the supremum being taken over all partitions $\{E_i\}$, of E. It turns out that $|\mu|$ actually is a positive measure on $\mathcal{E}$. Furthermore, $|\mu|(X) < \infty$. If μ is a positive measure, then $|\mu| = \mu$.

Let f be μ-integrable and, for $E \in \mathcal{E}$, let

$$\nu(E) = \int_E f d\mu . \tag{1.2.1}$$

Then, by Dunford-Schwartz [7,Th.20,p.114],

$$|\nu|(E) = \int_E |f| \, d|\mu|, \quad E \in \mathcal{E} \tag{1.2.2}$$

Let μ be a real measure (frequently called 'signed measure') on a σ-algebra $\mathcal{E}$. We define $|\mu|$ as before, and define

$$\mu^+ = \frac{1}{2}(|\mu| + \mu), \quad \mu^- = \frac{1}{2}(|\mu| - \mu) \tag{1.2.3}$$

Then μ^+ and μ^- are positive measures on $\mathcal{E}$, and are called positive and negative variations of μ, respectively.

$$\mu = \mu^+ - \mu^-, \quad |\mu| = \mu^+ + \mu^- \tag{1.2.4}$$

This representation of μ as the difference of the positive measures μ^+ and μ^- is called the Jordan decomposition of μ. If a function f is both μ^+- and μ^--integrable, then f is said to be μ-integrable, and we define

$$\int_E f d\mu = \int_E f d\mu^+ - \int_E f d\mu^- \quad (E \in \mathcal{E}) \qquad (1.2.5)$$

(see Munroe [23, p.184], Halmos [15, Sec.29 (7)])

Let μ be a complex measure on σ-algebra $\mathcal{E}$ in X. Then there is a measurable function h such that $|h(x)|=1$ for all $x\in X$ and such that

$$\mu(E) = \int_E h d|\mu| \quad (E \in \mathcal{E}) \qquad (1.2.6)$$

We may, therefore, define integration with respect to a complex measure μ by the formula

$$\int_E f d\mu = \int_E fh d|\mu| \quad (E \in \mathcal{E}) \qquad (1.2.7)$$

(see Rudin [33, Sec.6.18, eqn.(1)])

Let λ be a positive measure on a σ-algebra $\mathcal{E}$, and let μ be an arbitrary measure (positive or complex) on $\mathcal{E}$. μ is said to be absolutely continuous with respect to λ if to every $\varepsilon>0$ there corresponds a $\delta>0$ such that $|\mu(E)|<\varepsilon$ for all $E\in\mathcal{E}$ with $\lambda(E)<\delta$.

If X is a topological space, there exists a smallest σ-algebra $\mathcal{B}$ in X such that every open set in X belongs to $\mathcal{B}$. The members of $\mathcal{B}$ are called Borel sets of X. A measure μ defined on the σ-algebra of all

Borel sets in a locally compact Hausdorff space X is called a Borel measure on X. A complex Borel measure on X is called regular if for each $E \in \mathcal{B}$ and $\varepsilon > 0$ there exists a set F in $\mathcal{B}$ whose closure is contained in E and a set G in $\mathcal{B}$ whose interior contains E such that $|\mu(C)| < \varepsilon$ for every C in $\mathcal{B}$ with $C \subset G-F$.

The support of a complex function ψ on a topological space X is the closure of the set

$$\{x \mid \psi(x) \neq 0\} .$$

The collection of all continuous complex function on X whose support is compact is denoted by $C_c(X)$. This forms a normed linear space with addition, scalar multiplication and norm defined by

$$\begin{aligned} (\psi_1 + \psi_2)(x) &= \psi_1(x) + \psi_2(x) \\ (\alpha\psi)(x) &= \alpha\psi(x) \\ \|\psi\| &= \sup_{x \in X} |\psi(x)| \end{aligned} \qquad (1.2.8)$$

Let X be a locally compact Hausdorff space. If μ is a complex Borel measure on X then the mapping F defined by

$$F(\psi) = \int_X \psi \, d\mu \qquad (\psi \in C_c(X)) \qquad (1.2.9)$$

is a continuous linear functional on $C_c(X)$. Actually all continuous linear functionals on $C_c(X)$ are of this form. This follows from the Riesz Representation theorem which states that to each continuous linear functional F on $C_c(X)$, where X is a locally compact Hausdorff space, there

corresponds a unique complex regular Borel measure μ such that

$$F(\psi) = \int_X \psi \, d\mu \qquad (\psi \in C_c(X)) \tag{1.2.10}$$

Let f be a right continuous function of bounded variation on the open interval (a,b) which may be finite or infinite. We extend the domain of f to $[a,b]$ by defining $f(a) = f(b) = 0$. Then the set function defined by $\mu([a,d]) = f(d) - f(a)$ and $\mu((c,d]) = f(d)-f(c)$ for $a < c < d \leq b$ has a regular countably additive extension to the σ - algebra of all Borel sets in $[a,b]$. The restriction of this extension to the σ - algebra of Borel subsets of (a,b) is called the Radon or Borel-Steiltjes measure in (a,b) determined by the function f. Now let $\mathcal{B}^*$ consist of all sets of the form $E \cup N$ where $E \in \mathcal{B}$ and N is a subset of a set $M \in \mathcal{B}$ with $|\mu|(M) = 0$. Then $\mathcal{B}^*$ is a σ -algebra and if the domain of μ is extended to $\mathcal{B}^*$ by defining $\mu(E \cup N) = \mu(E)$, the extended function is countably additive on $\mathcal{B}^*$. The function μ with domain $\mathcal{B}^*$ is the Lebesgue - Stieltjes measure determined by the function f and the integral $\int_I g \, d\mu$ where $I = (a,b)$ is often written as $\int_a^b g(t)\, df(t)$. Also, since $|\mu|(J) = v(f,J)$ if J is any interval in I, the integral $\int_I g \, d|\mu|$ can be written as $\int_a^b g(t)\, dv_f(t)$. In the case where $f(t) = t$, μ is Lebesgue measure and the integral $\int_I g d\mu$ is written $\int_a^b g(t)dt$. When the set E is regarded as a variable, $\int_E g d\mu$ is called the indefinite integral of g with respect to μ. It is of bounded variation and absolutely continuous

with respect to μ .

Let f_k, $f \in BV([a,b])$ and $\lim_{k\to\infty} f_k(t) = f(t)$ on a dense subset of $[a,b]$ including the points a and b, and let $\lim_{j\to\infty} g_j = g$ a.e. with respect to LS-measure dh where h is a non-decreasing function such that

$$|\Delta f_k| \leq \Delta h, \qquad |\Delta f| \leq \Delta h$$

on every subinterval of $[a,b]$ (the symbol Δf on the interval $[\alpha,\beta]$, say, denotes $f(\beta)-f(\alpha)$). Then, by Graves [11, Th.27, Chap. XII], we have

$$\lim_{\substack{j\to\infty \\ k\to\infty}} \int_a^b g_j(t)df_k(t) = \int_a^b g(t)df(t) \tag{1.2.11}$$

if the indefinite integrals $\int g_j(t)dh$ be absolutely continuous with respect to LS-measure dh uniformly in j, and bounded uniformly.

Let $(X,\mathcal{E},\mu)$ be a measure space and f a complex valued μ-integrable function and

$$\lambda(E) = \int_E f\, d\mu \qquad (E \in \mathcal{E}) \tag{1.2.12}$$

Then, by Dunford-Schwartz [7, Cor.6, p.180], a function g on X to a Banach space B is λ-integrable iff fg is μ-integrable, and in this case we have

$$\int_E g d\lambda = \int_E fg d\mu \qquad (E \in \mathcal{E}). \tag{1.2.13}$$

1.3 DISTRIBUTIONS

Let Ω be an open set of R^n. By $C_c^\infty(\Omega)$ we denote the set of all complex functions $\in C_c(\Omega)$ which have partial derivatives of all orders $< \infty$. A classical example of a function $\in C_c^\infty(R^n)$ is

$$\begin{aligned} f(\xi) &= \exp\{(\|\xi\|^2-1)^{-1}\} \quad \text{for } \|\xi\| = \Big(\sum_{i=1}^{n} |\xi^i|^2\Big)^{1/2} < 1, \\ &= 0 \qquad\qquad \text{for } \|\xi\| \geq 1 \end{aligned} \tag{1.3.1}$$

$C_c^\infty(\Omega)$, structurized by equations (1.2.8), forms a linear manifold of $C_c(\Omega)$. A continuous linear functional defined on $C_c^\infty(\Omega)$ is called a distribution on Ω.

If μ is a complex Borel measure on Ω, then

$$F_\mu(\psi) = \int_\Omega \psi \, d\mu \qquad (\psi \in C_c^\infty(\Omega)) \tag{1.3.2}$$

defines a distribution on Ω. It follows from the paragraph containing equations (1.2.9) and (1.2.10) that the set of all complex regular Borel measures on Ω is, by $\mu \leftrightarrow F_\mu$, in one - one correspondence with the set of all distributions on Ω.

The set of all distributions on Ω is a linear space with addition and scalar multiplication defined by

$$\begin{aligned} (F_1+F_2)(\psi) &= F_1(\psi)+F_2(\psi), \\ (\alpha F)(\psi) &= \alpha F(\psi). \end{aligned} \tag{1.3.3}$$

This space of distributions on Ω, being the dual space of $C_c^\infty(\Omega)$, will be denoted by $C_c^\infty(\Omega)'$.

Let a complex function f defined a.e. on Ω be locally integrable on Ω with respect to the Lebesgue measure, in the sense that for any compact subset K of Ω, $\int_K |f(\xi)|\, d\xi < \infty$. Then

$$F_f(\psi) = \int_\Omega f(\xi)\psi(\xi)\, d\xi \ , \quad (\psi \in C_c^\infty(\Omega)) \tag{1.3.4}$$

defines a distribution F_f on Ω. It is proved (see [39, p.48]) that two distributions F_{f_1} and F_{f_2} are equal as functionals ($F_{f_1}(\psi) = F_{f_2}(\psi)$ for every $\psi \in C_c^\infty(\Omega)$) iff $f_1(\xi) = f_2(\xi)$ a.e. Hence the set of all locally integrable functions on Ω is, by $f \leftrightarrow F_f$, in one-one correspondence with a subset of $C_c^\infty(\Omega)'$ in such a way that (f_1 and f_2 being considered equivalent iff $f_1(\xi) = f_2(\xi)$ a.e.)

$$F_{f_1} + F_{f_2} = F_{f_1+f_2}, \ \alpha F_f = F_{\alpha f} \tag{1.3.5}$$

The derivative of a distribution F with respect to ξ^i, denoted by $D_i F$ or $\partial F/\partial \xi^i$, is defined by

$$D_i F(\psi) = -F(\partial\psi/\partial\xi^i) \quad (\psi \in C_c^\infty(\Omega)) \tag{1.3.6}$$

and is also a distribution on Ω. A distribution is infinitely differentiable in the sense of above definition.

Since a locally integrable function f on an open interval I of real line can be identified with the distribution F_f on I, $DF_f (\equiv dF_f/dt)$ will be denoted

by Df and called distributional derivative of f to distinguish from its ordinary derivative f' ($\equiv df/dt$). If f is absolutely continuous, then Df is the ordinary derivative f' (which is defined a.e.), f' being considered equivalent to the distribution $F_{f'}$. If f is of bounded variation then Df is the LS-measure df , df being considered as equivalent to the distribution F_{df} . Thus, for the Heaviside function H(t) defined by

$$H(t) = 1 \text{ or } 0 \text{ according as } t \geq 0 \text{ or } t < 0,$$

we have

$$DH \equiv DF_H = F_\delta \equiv \delta$$

where δ is the Dirac measure. In fact, we have, for any $\psi \in C_c^\infty(R)$,

$$DF_H(\psi) = -F_H(\psi') = -\int_{-\infty}^{\infty} H(t)\,\psi'(t)dt$$

$$= -\int_0^\infty \psi'(t)dt = -\left[\psi(t)\right]_0^\infty = \psi(0).$$

CHAPTER 2

FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS CONTAINING MEASURES

2.1 INTRODUCTION

The study of functional integro-differential equations containing measures is intiated in this chapter. The equation under consideration has first been converted into an equivalent integral equation (Theorem 1). The existence theorems and other properties in the present and subsequent chapters will be treated through this integral equation. The principle of contraction mapping is used for the proof of Local Existence and Uniqueness theorem (Theorem 2). In Local Existence theorem (Theorem 3) a method of successive approximations is used.

2.2 FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS CONTAINING MEASURES

Let S be a domain (an open connected set) in R^n. The set of all functions in $BV(I)_n$ with values in S will be denoted by $BV(I,S)$. Let α,β and t_o be numbers such that

$$-\infty \leq \alpha < t_o < \beta \leq \infty \qquad (2.2.1)$$

In what follows, the interval $[\alpha,t_o]$ will be understood to be $(-\infty,t_o]$ in case $\alpha = -\infty$; similarly

CHAPTER 2

FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS CONTAINING MEASURES

2.1 INTRODUCTION

The study of functional integro-differential equations containing measures is intiated in this chapter. The equation under consideration has first been converted into an equivalent integral equation (Theorem 1). The existence theorems and other properties in the present and subsequent chapters will be treated through this integral equation. The principle of contraction mapping is used for the proof of Local Existence and Uniqueness theorem (Theorem 2). In Local Existence theorem (Theorem 3) a method of successive approximations is used.

2.2 FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS CONTAINING MEASURES

Let S be a domain (an open connected set) in R^n. The set of all functions in $BV(I)_n$ with values in S will be denoted by $BV(I,S)$. Let α, β and t_0 be numbers such that

$$-\infty \leq \alpha < t_0 < \beta \leq \infty \qquad (2.2.1)$$

In what follows, the interval $[\alpha, t_0]$ will be understood to be $(-\infty, t_0]$ in case $\alpha = -\infty$; similarly

if $\beta = \infty$ the interval $[t_o, \beta]$ will mean $[t_o, \infty)$. Let p and q be two real functions defined on $[t_o, \beta]$ and satisfying

$$\alpha \leq p(t) \leq q(t) \leq t \tag{2.2.2}$$

for each $t \in [t_o, \beta]$. We define the interval

$$I_t = [p(t), q(t)] \tag{2.2.3}$$

Given $x \in BV([\alpha, t])$ where $t \in [t_o, \beta]$, we define the mapping x_t on I_t by

$$x_t(s) = x(s) \quad , s \in I_t ; \tag{2.2.4}$$

i.e., x_t is the restriction of the function x on the interval I_t. We further define

$$E_t = \{(t, x_t) \mid x \in BV([\alpha, t], S)\}, \quad t \in [t_o, \beta] \tag{2.2.5}$$

and

$$E = \bigcup_{t \in [t_o, \beta]} E_t \tag{2.2.6}$$

Let the following be given :

(i) a vector functional f defined on E with values in R^n,

(ii) a matrix functional G defined on E with values in the set $\mathcal{M}$ of all n x m matrices,

(iii) a vector function H defined on S with values in R^m,

(iv) a right continuous function $u \in BV([t_o, \beta])_m$,

(v) a right continuous function $\lambda \in BV([t_o, \beta])_m$.

We assume that for each given $x \in BV([\alpha,T],S)$, $t_o \leq T \leq \beta$, $H(x(t))$ is measurable with respect to the LS-measure $d\lambda$ on the interval $[t_o,T]$. Consider the integro-differential system

$$Dx = f(t,x_t) + G(t, x_t)\ Du + + \int_{t_o}^{t} H\ (x(t+t_o-\tau)d\lambda(\tau), \quad t>t_o \tag{2.2.7}$$

where the operations of differentiation are to be understood in the sense of distributional derivatives with respect to the real variable t. Since the distributional derivative of a function of bounded variation can always be identified with a LS-measure, we shall call (2.2.7) a measure integro-differential equation.

DEFINITION 1 : A function $x=x(.;t_o,\varphi)$ is called a solution of (2.2.7) on an interval I, $[\alpha,t_o] \subset I \subset [\alpha,\beta]$, with the initial function $\varphi \in BV([\alpha,t_o],S)$, if

(i) $x \in BV(I,S)$

(ii) $x(t) = \varphi(t)$ for $t \in [\alpha,t_o]$

(iii) x is right continuous on $I \cap [t_o,\beta]$

(iv) the distributional derivative Dx of x on (t_o,T) for any arbitrary $T \in I \cap (t_o,\beta]$ satisfies (2.2.7).

Assume that for each given $x \in BV([\alpha,T],S)$, $t_o \leq T \leq \beta$, $f(t,x_t)$ is Lebesgue measurable, $G(t,x_t)$ and $H(x(t))$ are integrable with respect to the LS-measures

du and $d\lambda$, respectively, on the interval $[t_o, T']$.

Consider now the integral equation

$$x(t) = \begin{cases} \varphi(t) \text{ for } t \in [\alpha, t_o] \\ \varphi(t_o) + \int_{t_o}^{t} f(s, x_s) ds + \int_{t_o}^{t} G(s, x_s) du(s) \\ \qquad + \int_{t_o}^{t} \left\{ \int_{t_o}^{s} H(x(s+t_o-\tau)) d\lambda(\tau) \right\} ds \\ \qquad\qquad \text{for } t > t_o \end{cases} \tag{2.2.8}$$

DEFINITION 2 : A function x is called a solution of (2.2.8) on an interval I, $[\alpha, t_o] \subset I \subset [\alpha, \beta]$ if

(i) $x \in BV(I, S)$

(ii) x satisfies (2.2.8) on I.

We shall now show that the equations (2.2.7) and (2.2.8) are equivalent in the sense that a solution of (2.2.7) with initial function φ is a solution of (2.2.8) and conversely. In the proof the following lemma will be used.

LEMMA : If g is a function integrable with respect to μ, and F is a distribution on Ω given by

$$F(\psi) = \int_{\Omega} \psi \, d\mu, \quad \psi \in C_c^{\infty}(\Omega), \tag{2.2.9}$$

then the product gF defined by

$$(gF)(\psi) = \int_{\Omega} g \psi \, d\mu, \; \psi \in C_c^{\infty}(\Omega) \tag{2.2.10}$$

is also a distribution on Ω.

PROOF : Since $\psi \in C_c^{\infty}(\Omega)$, it is bounded and μ-measurable and g is given to be μ-integrable. Therefore, by Munroe

hand side of (2.2.10) is meaningful. gF defined by (2.2.10) is obviously a linear functional on $C_c^{\infty}(\Omega)$. Furthermore,

$$|(gF)(\psi)| \leq \int_{\Omega} |g||\psi| d|\mu|$$

$$\leq \|\psi\| \int_{\Omega} |g| d|\mu|$$

Therefore,

$$\|gF\| = \sup \{ |(gF)(\psi)| : \|\psi\| \leq 1 \}$$
$$\leq \int_{\Omega} |g| d|\mu| < \infty ,$$

since μ - integrability of g implies $|\mu|$ - integrability of $|g|$, by Dunford-Schwartz [7, Lemma 18, p.113]. Thus gF is bounded (and hence continuous) linear functional on $C_c^{\infty}(\Omega)$, and is, therefore, a distribution on Ω.

<u>Theorem 1</u> :

x is a solution of (2.2.7) with initial function $\varphi \in BV([\alpha, t_0], S)$ if and only if it is a solution of (2.2.8).

PROOF : Let x be a solution of (2.2.8) on an interval I . Then conditions (i) and (ii) of Definition 1 are satisfied. The right continuity of u implies that the integral $\int_{t_0}^{t} G(s,x_s)du(s)$ is also right continuous function of t. The other two integrals in the second equation of (2.2.8) are absolutely continuous (and hence continuous) functions of t. Thus x is right continuous satisfying the condition (iii) of Definition 1. To show that the condition (iv) of Definition 1 is also satisfied, let F^i be the distribution on (t_0,T) to be identified with the ith component x^i of x. Then

$$F^i(\psi) = \int_J \Big(\varphi^i(t_o)+\int_{t_o}^{t} f^i(s,x_s)ds+\int_{t_o}^{t}[G(s,x_s)du(s)]^i$$
$$+\int_{t_o}^{t}\Big\{\int_{t_o}^{s}[H(x(s+t_o-\tau)d\lambda(\tau)]^i\Big\}ds\Big)\psi(t)dt \qquad (2.2.11)$$

for all $\psi \in C_c^\infty(J)$ where $J = (t_o,T)$. The distributional derivative is

$$DF^i(\psi) = -F^i(\psi')$$
$$= -\int_J[\varphi^i(t_o)+\int_{t_o}^{t} f^i(s,x_s)ds+\int_{t_o}^{t}\Big(\sum_{j=1}^{m} G_j^i(s,x_s)du^j(s)\Big)$$
$$+\int_{t_o}^{t}\Big\{\int_{t_o}^{s}\Big(\sum_{j=1}^{m} H_j^i(x(s+t_o-\tau))d\lambda^j(\tau)\Big)\Big\}ds]\psi'(t)dt$$

$$(2.2.12)$$

where $G_j^i(t,x_t)$ and $H_j^i(\xi)$ are i,jth elements of $G(t,x_t)$ and $H(\xi)$ respectively, and $u^j(t)$ and $\lambda^j(t)$ are jth components of $u(t)$ and $\lambda(t)$ respectively.

Integration by parts yields

$$\int_J[\varphi^i(t_o)+\int_{t_o}^{t} f^i(s,x_s)ds+\int_{t_o}^{t}\Big\{\int_{t_o}^{s}\sum_{j=1}^{m}(H_j^i(x(s+t_o-\tau))d\lambda^j(\tau))\Big\}ds]\psi'(t)dt$$

$$= -\int_J \psi(t)\Big[f^i(s,x_s)$$
$$+\int_{t_o}^{t}\sum_{j=1}^{m} H_j^i(x(t+t_o-\tau))d\lambda^j(\tau)\Big]dt, \qquad (2.2.13)$$

since $\psi(t_o) = \psi(T) = 0$.

The function $g(t) = \int_{t_o}^{t} G_j^i(s,x_s)du^j(s)$ is right continuous and is of bounded variation on the interval $J=(t_o,T)$. We have

$$\int_J g(t)\psi'(t)dt = \int_{(t_o,T)} g(t)\,d\psi(t)$$
$$=\int_{(t_o,T]} g(t)d\psi(t) - \int_{\{T\}} g(t)d\psi(t).$$

But $\int_{\{T\}} g(t)\, d\psi(t) = 0$, since ψ is continuous ; and

$$\int_{(t_o,T]} g(t)d\psi(t) = g(T)\psi(T)-g(t_o)\psi(t_o)$$
$$-\int_{(t_o,T]} \psi(t)dg(t), \text{ by Munroe[23, Ex.n, p. 185]}$$
$$=-\int_{(t_o,T]}\psi(t)dg(t), \text{since } \psi(t_o)=\psi(T)=0$$
$$=-\int_{(t_o,T)} \psi(t)dg(t)-\int_{\{T\}}\psi(t)dg(t)$$
$$=-\int_J \psi(t)dg(t)- \psi(T)\ (g(T)-g(T-))$$
$$=-\int_J \psi(t)dg(t), \quad \text{since } \psi(T)=0.$$

Therefore,

$$\int_J g(t)\psi'(t)dt = - \int_J \psi(t)dg(t)$$

That is

$$\int_J\left\{ \int_{t_o}^t G_j^i(s,x_s)du^j(s)\right\} \psi'(t)dt$$
$$= - \int_J\psi(t)d\left\{ \int_{t_o}^t G_j^i\ (s,x_s)du^j(s)\right\}$$
$$= - \int_J\psi(t)\ G_j^i(s,x_s)du^j(s)\ , \text{ by } (1.2.13)$$

Summation over j in the above equation yields

$$\int_J\left\{ \int_{t_o}^t \left(\sum_{j=1}^m G_j^i\ (s,x_s)\ du^j(s)\right)\right\} \psi'(t)dt$$
$$= -\int_J \psi(t)\ \left(\sum_{j=1}^m G_j^i\ (s,x_s)\ du^j(s)\right) \tag{2.2.14}$$

From (2.2.12), (2.2.13) and (2.2.14), we obtain

$$DF^i(\psi) = \int_J \psi(t)\ f^i(t,x_t)\ dt$$
$$+\int_J \psi(t)\left\{ \int_{t_o}^t[H(x(t+t_o-\tau))d\lambda(\tau)]^i\right\}\ dt$$
$$+\int_J \psi(t)\ [G(t,x_t)du(t)]^i. \tag{2.2.15}$$

By the above Lemma, the last continuous linear functional

are identified with $f^i(t,x_t)$ and $\int_{t_o}^{t}[H(x(t+t_o-\tau))d\lambda(\tau)]^i$ respectively. Thus the distributional derivative $DF(\psi)$ is identified with

$$f(t,x_t) + G(t,x_t)Du + \int_{t_o}^{t} H(x(t+t_o-\tau))d\lambda(\tau)$$

Hence x is a solution of (2.2.7).

Conversely, let x be a solution of (2.2.7) on the interval I with initial function $\varphi \in BV([\gamma,t_o],S)$. Then for $J=(t_o,T)$, where T is an arbitrary point in $I\cap(t_o,\beta]$, we have

$$\begin{aligned}\int_J \psi(t)dx^i(t) &= \int_J \psi(t)f^i(t,x_t)dt \\ &+ \int_J \psi(t)[G(t,x_t)du(t)]^i \qquad (2.2.16)\\ &+ \int_J \psi(t)\{\int_{t_o}^{t}[H(x(t+t_o-\tau)d\lambda(\tau)]^i\}dt\end{aligned}$$

$$i=1,2,\ldots,n$$

for all $\psi \in C_c^\infty(J)$. Integrating the left hand side of (2.2.16) by parts and using (2.2.13) and (2.2.14) we obtain

$$\begin{aligned}&\int_J \psi'(t)(x^i(t)-\varphi^i(t_o))dt \\ &= \int_J \psi'(t)\{\int_{t_o}^{t} f^i(s,x_s)ds + \int_{t_o}^{t}[G(s,x_s)du(s)]^i \\ &+ \int_{t_o}^{t}(\int_{t_o}^{s}[H(x(s+t_o-\tau))d\lambda(\tau)]^i)\ ds\}\ dt\end{aligned}$$

Therefore,

$$\begin{aligned}x^i(t)=\varphi^i(t_o) &+ \int_{t_o}^{t} f^i(s,x_s)ds+\int_{t_o}^{t}[G(s,x_s)du(s)]^i \\ &+ \int_{t_o}^{t}\{\int_{t_o}^{s}[H\ (x(s+t_o-\tau))d\lambda(\tau)]^i\}ds \qquad (2.2.17)\end{aligned}$$

a.e. in J. But, since x^i is right continuous, x being solution of (2.2.7), and since the right hand side of (2.2.17) is a right continuous function of t, equality holds everywhere in J in (2.2.17). Hence x is a solution of (2.2.8). This completes the proof of Theorem 1.

2.3 EXISTENCE AND UNIQUENESS OF SOLUTIONS

We first enumerate the hypotheses which will be used in this section (and also in Chapter 3). In these hypotheses T will be assumed to be fixed in $(t_o, \beta]$.

H_o : For each given $x \in BV([\alpha, T], S)$, $f(t, x_t)$ is Lebesgue measurable, $G(t, x_t)$ and $H(x(t))$ are integrable with respect to the LS-measures du and $d\lambda$, respectively, on the interval $[t_o, T]$.

H_1 : $f(t, x_t)$ is locally Lipschitzian with respect to x for $t \in [t_o, T]$; i.e. for any $t_1 \in [t_o, T]$ and every $c_1 < \infty$, there exists $c_2 = c_2(t_1, c_1)$ such that $\| x \|_{[\alpha, t_1]} \leq c_1$, $\| y \|_{[\alpha, t_1]} \leq c_1$ imply

$$|f(t, x_t) - f(t, y_t)| \leq c_2 \| x-y \|_{I_t}$$

for every $t \in [t_o, t_1]$.

H_2 : $\lim_{k \to \infty} x^{(k)}(s) = x^*(s)$ for each $s \in [\alpha, T]$ implies

$$\lim_{k \to \infty} f(t, x_t^{(k)}) = f(t, x_t^*), \quad t \in [t_o, T]$$

H_3 : There exists a real function r defined and integrable with respect to Lebesgue measure on $[t_o, T]$ such that

$$|f(t, x_t)| \leq r(t)$$

uniformly with respect to $x \in BV([\alpha, T], S)$.

H_4 : $G(t, x_t)$ is locally Lipschitzian with respect to x for $t \in [t_o, T]$.

H_5 : $\lim_{k\to\infty} x^{(k)}(t) = x^*(t)$ for each $t \in [\alpha, T]$

and $\lim_{k\to\infty} t_k = t^*$ imply

$$\lim_{k\to\infty} G(t_k, x_{t_k}^{(k)}) = G(t^*, x_{t^*}^*), \quad (t_k \in [t_0, T])$$

H_6 : There exists a real function w defined and integrable with respect to LS-measure dv_u on $[t_o, T]$ such that

$$|G(t, x_t)| \leq w(t)$$

uniformly with respect to $x \in BV([\alpha, T], S)$

H_7 : w in H_6 is non-decreasing.

H_8 : H is locally Lipschitzian on S ; i.e. for each $c_1 < \infty$ there exists $c_2 = c_2(c_1)$ such that $\xi, \eta \in S$, $|\xi| \leq c_1$, $|\eta| \leq c_1$ imply

$$|H(\xi) - H(\eta)| \leq c_2 |\xi - \eta|.$$

H_9 : H is continuous on S.

H_{10} : There exists a real function z, defined and integrable with respect to Lebesgue measure on $[t_o, T]$, such that

$$|H(x(t))| \leq z(t)$$

uniformly with respect to $x \in BV([\alpha, T], S)$.

Theorem 2 (Local Existence and Uniqueness) :

Let the hypotheses H_o , H_1, H_3, H_4, H_6, H_8 and H_{10} be satisfied. Then there exists a unique solution of (2.2.7) on an interval $[\alpha, t_o + a]$, $a>0$, with a given intial function $\varphi \in BV([\alpha, t_o], S)$.

PROOF : For each $t \in [t_o, T]$, we shall denote by Q_t the set of all functions x with the properties

(i) $x \in BV([\alpha, t])_n$

(ii) $x(s) = \varphi(s)$ for $s \in [\alpha, t_o]$

(iii) $v(x, [t_o, t]) \leq b$ where $b>0$

Suppose that $Q_t \subset BV([\alpha, t], S)$. This is always possible if b is suitably chosen.

Choose a, $0 < a \leq T - t_o$, such that

$$\int_{t_o}^{t_o+a} r(s)ds + \int_{t_o}^{t_o+a} w(s)dv_u(s)$$

$$+v(\lambda, [t_o, t_o+a]) \int_{t_o}^{t_o+a} z(s)ds \leq b. \qquad (2.3.1)$$

Since $\int_{t_o}^{t} r(s)ds$ and $\int_{t_o}^{t} z(s)ds$ are continuous functions of t and $\int_{t_o}^{t} w(s)dv_u(s)$ is right continuous function of t, it is possible to choose such an a.

Now consider $Q_{t_o + a}$ which forms a complete metric space. Let A be the mapping defined on Q_{t_o+a} by

$$(Ax)(t) = \begin{cases} \varphi(t) \text{ for } t \in [\alpha, t_o] \\ \varphi(t_o) + \int_{t_o}^{t} f(s, x_s)ds + \int_{t_o}^{t} G(s, x_s)du(s) \\ \quad + \int_{t_o}^{t} \left\{ \int_{t_o}^{s} H(x(s+t_o-\tau))\, d\lambda(\tau) \right\} ds \\ \qquad \text{for } t \in (t_o, t_o+a] \end{cases} \qquad (2.3.2)$$

Since each of the integrals on the right of (2.3.2) is a function in t which is of bounded variation on $[t_o, t_o+a]$, it follows that $Ax \in BV([\alpha, t_o+a])_n$. Furthermore,

$$v(Ax,[t_o,t_o+a]) \leq v(\int_{t_o}^{t} f(s,x_s)ds, [t_o,t_o+a])$$
$$+ v(\int_{t_o}^{t} G(s,x_s)du(s), [t_o,t_o+a])$$
$$+ v(\int_{t_o}^{t} \left\{ \int_{t_o}^{t} H(x(s+t_o-\tau))d\lambda(\tau) \right\} ds, [t_o,t_o+a])$$
$$\leq \int_{t_o}^{t_o+a} |f(s,x_s)|ds + \int_{t_o}^{t_o+a} |G(s,x_s)|dv_u(s)$$
$$+\int_{t_o}^{t_o+a} \left| \int_{t_o}^{s} H(x(s+t_o-\tau))d\lambda(\tau) \right| ds \tag{2.3.3}$$

But

$$\int_{t_o}^{t_o+a} \left| \int_{t_o}^{s} H(x(s+t_o-\tau))d\lambda(\tau) \right| ds$$
$$\leq \int_{t_o}^{t_o+a} \left\{ \int_{t_o}^{s} |H(x(s+t_o-\tau))| \, dv_\lambda(\tau) \right\} ds$$
$$\leq \int_{t_o}^{t_o+a} \left\{ \int_{t_o}^{s} z(s+t_o-\tau)dv_\lambda(\tau) \right\} ds, \text{ by } H_{10}$$
$$= \int_{t_o}^{t_o+a} \left\{ \int_{\tau}^{t_o+a} z(s+t_o-\tau)ds \right\} dv_\lambda(\tau),$$
by Fubini's theorem
$$= \int_{t_o}^{t_o+a} \left\{ \int_{t_o}^{2t_o+a-\tau} z(s)ds \right\} dv_\lambda(\tau)$$
$$\leq \int_{t_o}^{t_o+a} \left\{ \int_{t_o}^{t_o+a} z(s)ds \right\} dv_\lambda(\tau)$$
$$= v(\lambda, [t_o,t_o+a]) \int_{t_o}^{t_o+a} z(s)ds \tag{2.3.4}$$

From (2.3.3), (2.3.4), H_3 and H_6, we obtain

$$v(Ax, [t_o, t_o+a]) \leq \int_{t_o}^{t_o+a} r(s)ds + \int_{t_o}^{t_o+a} w(s)dv_u(s)$$

$$+v(\lambda, [t_o, t_o+a]) \int_{t_o}^{t_o+a} z(s)ds$$

$$\leq b, \text{ by } (2.3.1) \tag{2.3.5}$$

Thus A maps Q_{t_o+a} into itself. We shall show that A is a contraction.

We have

$$\| Ax-Ay \|_{[\alpha, t_o+a]}$$
$$\leq \| \int_{t_o}^{t} [f(s,x_s)-f(s,y_s)] ds \|_{[t_o, t_o+a]}$$
$$+ \| \int_{t_o}^{t} [G(s,x_s)-G(s,y_s)] du(s) \|_{[t_o, t_o+a]}$$
$$+ \| \int_{t_o}^{t} \left\{ \int_{t_o}^{s} [H(x(s+t_o-\tau))-H(y(s+t_o-\tau))] d\lambda(\tau) \right\} ds \|_{[t_o, t_o+a]}$$
$$\leq \int_{t_o}^{t_o+a} |f(s,x_s)-f(s,y_s)| ds$$
$$+\int_{t_o}^{t_o+a} |G(s,x_s)-G(s,y_s)| dv_u(s)$$
$$+\int_{t_o}^{t_o+a} \left\{ \int_{t_o}^{s} |H(x(s+t_o-\tau))-H(y(s+t_o-\tau))| dv_\lambda(\tau) \right\} ds \tag{2.3.6}$$

We have, for every $x \in Q_{t_o+a}$,

$$\| x \|_{[\alpha, t_o+a]} = \| \varphi \|_{[\alpha, t_o]} + v(x, [t_o, t_o+a])$$
$$\leq \| \varphi \|_{[\alpha, t_o]} + b = c_1, \text{ say.}$$

Also,

$$|x(t)| \leq \|x\|_{[\alpha,t_o+a]} \leq c_1$$

Therefore, by H_1, H_4 and H_8, there exist constants $c_2 = c_2(c_1), c_3 = c_3(c_1), c_4 = c_4(c_1)$ such that

$$|f(s,x_s)-f(s,y_s)| \leq c_2\|x-y\|_{I_s} \leq c_2\|x-y\|_{[\alpha,s]} \tag{2.3.7}$$

$$|G(s,x_s)-G(s,y_s)| \leq c_3\|x-y\|_{I_s} \leq c_3\|x-y\|_{[\alpha,s]} \tag{2.3.8}$$

$$\begin{aligned} &|H(x(s+t_o-\tau)) - H(y(s+t_o-\tau))| \\ &\leq c_4|x(s+t_o-\tau)-y(s+t_o-\tau)| \\ &\leq c_4\|x-y\|_{[\alpha,s]} \end{aligned} \tag{2.3.9}$$

From (2.3.6), (2.3.7), (2.3.8) and (2.3.9), we obtain

$$\begin{aligned} &\|Ax-Ay\|_{[\alpha,t_o+a]} \\ &\leq \int_{t_o}^{t_o+a} c_2\|x-y\|_{[\alpha,s]}ds + \int_{t_o}^{t_o+a} c_3\|x-y\|_{[\alpha,s]}dv_u(s) \\ &+ \int_{t_o}^{t_o+a} c_4\left\{\int_{t_o}^{s} \|x-y\|_{[\alpha,s]}dv_\lambda(\tau)\right\} ds \\ &\leq \Big\{ac_2 + c_3v(u, [t_o,t_o+a]) + \\ &\qquad +ac_4v(\lambda, [t_o,t_o+a]\Big\}\|x-y\|_{[\alpha,t_o+a]} \end{aligned} \tag{2.3.10}$$

Due to right continuity of u and hence of v_u, 'a' can be chosen such that

$$\begin{aligned} &ac_2 + c_3v(u, [t_o,t_o+a]) \\ &\qquad + ac_4v(\lambda, [t_o,t_o+a]) < 1 \end{aligned} \tag{2.3.11}$$

and then A, by (2.3.10), is a contraction. Hence by the principle of contraction mapping there is a unique fixed point. This completes the proof.

The next theorem shows that the Lipschitz conditions are not necessary for the existence of solutions of (2.2.7).

<u>Theorem 3</u> (Local Existence) :

Let the hypotheses H_o, H_2, H_3, H_5, H_6, H_7, H_9 and H_{10} be satisfied. Then there exists a solution of (2.2.7) on an interval $[\alpha, t_o+a]$, $a>0$, with a given initial function $\varphi \in BV([\alpha, t_o], \mathcal{E})$.

PROOF : Define Q_t as in the proof of Theorem 2 and choose $a>0$ so as to satisfy (2.3.1).

Now define

$$\overline{\varphi}(t) = \begin{cases} \varphi(t) \text{ for } t \in [\alpha, t_o] \\ \varphi(t_o) \text{ for } t \in (t_o, t_o+a] \end{cases} \tag{2.3.12}$$

and

$$x^{(k)}(t) = \begin{cases} \overline{\varphi}(t) \text{ for } t \in [\alpha, t_o+a/k] \\ \varphi(t_o) + \int_{t_o}^{t-a/k} f(s, x_s^{(k)})\, ds \\ \quad + \int_{t_o+a/k}^{t} G(s-a/k, x_{s-a/k}^{(k)})\, du(s) \\ \quad + \int_{t_o}^{t-a/k} \left\{ \int_{t_o}^{s} H(x^{(k)}(s+t_o-\tau))\, d\lambda(\tau) \right\} ds \\ \qquad \text{for } t \in (t_o+a/k, t_o+a] \end{cases} \tag{2.3.13}$$

$k = 1, 2, \ldots.$

For any $k \geq 2$ the first expression in (2.3.13) defines $x^{(k)}$ on $[\alpha, t_o+a/k]$ and then the second expression of (2.3.13) defines $x^{(k)}$ on $(t_o+a/k, t_o+2a/k]$. Let us assume that $x^{(k)}$ is defined on $[\alpha, t_o+ja/k]$ for $1<j<k$. Then the second expression of (2.3.13) defines $x^{(k)}$ on

$(t_o+ja/k, t_o+(j+1)a/k]$. $x^{(k)}$ is thus defined on $[\tau, t_o+a]$. Now,

$$v(x^{(k)}, [t_o, t_o+a])$$

$$= v(x^{(k)}, [t_o+a/k,\ t_o+a])$$

$$\leq \int_{t_o}^{t_o+(k-1)a/k} |f(s, x_s^{(k)})|\,ds$$

$$+ \int_{t_o+a/k}^{t_o+a} |G(s-a/k, x_{s-a/k}^{(k)})|\,dv_u(s)$$

$$+ \int_{t_o}^{t_o+(k-1)a/k} \left\{\int_{t_o}^{s} |H(x^{(k)}(s+t_o-\tau))|\,dv_\lambda(\tau)\right\} ds$$

$$\leq \int_{t_o}^{t_o+(k-1)a/k} r(s)\,ds + \int_{t_o+a/k}^{t_o+a} w(s-a/k)\,dv_u(s)$$

$$+ \int_{t_o}^{t_o+(k-1)a/k} \left\{\int_{t_o}^{s} z(s+t_o-\tau)\,dv_\lambda(\tau)\right\} ds$$

$$\leq \int_{t_o}^{t_o+a} r(s)\,ds + \int_{t_o}^{t_o+a} w(s)\,dv_u(s)$$

$$+ \int_{t_o}^{t_o+a} \left\{\int_{t_o}^{s} z(s+t_o-\tau)\,dv_\lambda(\tau)\right\} ds \tag{2.3.14}$$

since $w(s-a/k) \leq w(s)$, by H_7, and $r(s)$, $w(s)$ and

$\int_{t_o}^{s} z(s+t_o-\tau)\,dv_\lambda(\tau)$ are all positive functions of s.

But

$$\int_{t_o}^{t_o+a} \left\{\int_{t_o}^{s} z(s+t_o-\tau)\,dv_\lambda(\tau)\right\} ds$$

$$\leq v(\lambda, [t_o, t_o+a]) \int_{t_o}^{t_o+a} z(s)\,ds\ , \tag{2.3.15}$$

as in (2.3.4)

By (2.3.14), (2.3.15) and (2.3.1), we obtain

$$v(x^{(k)}, [t_o, t_o+a]) \leq b, \quad k=1,2,\ldots \tag{2.3.16}$$

Thus,

$$x^{(k)} \in Q_{t_o+a}\ , \quad k = 1, 2, \ldots.. \tag{2.3.17}$$

and hence $x^{(k)}$ are of uniformly bounded variation and and also are uniformly bounded. By Helly's selection principle (see (1.1.3)), there exists a subsequence $x^{(k_j)}$ and a function x^* of bounded variation such that

$$\lim_{j \to \infty} x^{(k_j)}(t) = x^*(t) \tag{2.3.18}$$

for each $t \in [\alpha, t_o+a]$, and

$$v(x^*, [\alpha, t_o+a]) \leq \varliminf_{j\to\infty} v(x^{(k_j)}, [\alpha, t_o+a])$$

$$\leq b, \text{ by } (2.3.16)$$

Obviously, $x^*(t) = \varphi(t)$ for $t \in [\alpha, t_o]$. Hence $x^* \in Q_{t_o+a} \subset BV([\alpha, t_o+a], S)$.

For $t \in [t_o, \beta]$, x_t is defined by (2.2.4). For $t \in [\alpha, t_o)$, let x_t denote the restriction of the function x on $[\alpha, t]$. Also, let

$$E_o = \{(t, x_t) \mid t \in [\alpha, t_o),\ x \in BV[\alpha, t_o]\} \tag{2.3.19}$$

Define $\tilde{G}$ on $E \cup E_o$ by

$$\tilde{G}(t, x_t) = \begin{cases} G(t_o, x_{t_o}) & \text{for } t \in [\alpha, t_o] \\ G(t, x_t) & \text{for } t \in (t_o, \beta] \end{cases} \tag{2.3.20}$$

Then, we have, by (2.3.13),

$$x^{(k_j)}(t) = \begin{cases} \tilde{\varphi}(t) \text{ for } t \in \langle, t_o+a/k_j \\ \varphi(t_o)+\int_{t_o}^{t} f(s,x_s^{(k_j)})ds-\int_{t-a/k_j}^{t} f(s,x_s^{(k_j)})ds \\ +\int_{t_o}^{t} \tilde{G}(s-a/k_j,x_{s-a/k_j}^{(k_j)})\, du(s) \\ -\int_{t_o}^{t_o+a/k_j} \tilde{G}(s-a/k_j, x_{s-a/k_j}^{(k_j)})du(s) \\ +\int_{t_o}^{t} \left\{ \int_{t_o}^{s} H(x^{(k_j)}(s+t_o-\tau))d\lambda(\tau) \right\} ds \\ -\int_{t-a/k_j}^{t} \left\{ \int_{t_o}^{s} H(x^{(k_j)}(s+t_o-\tau))d\lambda(\tau) \right\} ds \\ \qquad \text{for } t \in (t_o+a/k_j,\ t_o+a] \end{cases} \tag{2.3.21}$$

Now,

$$\lim_{j\to\infty} \int_{t_o}^{t} f(s,x_s^{(k_j)})\, ds = \int_{t_o}^{t} f(s,x_s^{*})ds \tag{2.3.22}$$

by (2.3.18), H_2 and H_3, using Lebesgue's dominated convergence theorem.

Let du^+ and du^- be positive and negative variations of LS-measure du. Then we have, by (1.2.5),

$$\begin{aligned}\int_{t_o}^{t} \tilde{G}(s-a/k_j, x_{s-a/k_j}^{(k_j)})du(s) \\ = \int_{t_o}^{t} \tilde{G}(s-a/k_j,x_{s-a/k_j}^{(k_j)})du^+(s) \\ -\int_{t_o}^{t} \tilde{G}(s-a/k_j,x_{s-a/k_j}^{(k_j)})du^-(s)\end{aligned} \tag{2.3.23}$$

Now take limits as $j\to\infty$. Due to (2.3.18), H_5, H_6 and (2.3.20), Lebesgue's dominated convergence theorem

(2.3.23) to obtain

$$\lim_{j\to\infty} \int_{t_o}^{t} \widetilde{G}(s-a/k_j, x^{(k_j)}_{s-a/k_j}) du(s)$$

$$= \int_{t_o}^{t} \widetilde{G}(s,x^*_s)\, du^+(s) - \int_{t_o}^{t} \widetilde{G}(s,x^*_s)\, du^-(s)$$

$$= \int_{t_o}^{t} \widetilde{G}(s,x^*_s) du(s) = \int_{t_o}^{t} G(s,x^*_s) du(s) \qquad (2.3.24)$$

Similarly,

$$\lim_{j\to\infty} \int_{t_o}^{t} H(x^{(k_j)}(s+t_o-\tau)) d\lambda(\tau)$$

$$= \int_{t_o}^{t} H\, (x^*(s+t_o-\tau))\; d\lambda(\tau). \qquad (2.3.25)$$

Also,

$$\left| \int_{t_o}^{t} H(x^{(k_j)}(s+t_o-\tau)) d\lambda(\tau) \right|$$

$$\leq \int_{t_o}^{t} z(s+t_o-\tau))\; dv_\lambda(\tau). \qquad (2.3.26)$$

From (2.3.25) and (2.3.26) we obtain, by using Lebesgue's dominated convergence theorem,

$$\lim_{j\to\infty} \int_{t_o}^{t} \left\{ \int_{t_o}^{s} H(x^{(k_j)}(s+t_o-\tau)) d\lambda(\tau) \right\} ds$$

$$= \int_{t_o}^{t} \left\{ \int_{t_o}^{s} H(x^*(s+t_o-\tau))\; d\lambda(\tau) \right\}\; ds \qquad (2.3.27)$$

Also,

$$\lim_{j\to\infty} \int_{t_o}^{t_o+a/k_j} \widetilde{G}\,(s-a/k_j,\; x^{(k_j)}_{s-a/k_j}) du(s) = 0, \qquad (2.3.28)$$

since

$$\left| \int_{t_o}^{t_o+a/k_j} \tilde{G}\,(s-a/k_j,\ x^{(k_j)}_{s-a/k_j})\, du(s) \right|$$

$$\leq \left| G(t_o,\varphi_{t_o}) \right| v(u, [t_o, t_o+a/k_j])$$

$$\to 0 \quad \text{as } j \to \infty .$$

Further,

$$\left| \int_{t-a/k_j}^{t} f(s, x_s^{(k_j)})\, ds \right| \leq \int_{t-a/k_j}^{t} r(s) ds$$

$$\to 0 \text{ as } j \to \infty \qquad (2.3.29)$$

and

$$\left| \int_{t-a/k_j}^{t} \left\{ \int_{t_o}^{s} H(x^{(k_j)}(s+t_o-\tau))\, d\lambda(\tau) \right\} ds \right|$$

$$\leq \int_{t-a/k_j}^{t} \left\{ \int_{t_o}^{s} z(s+t_o-\tau) dv_\lambda(\tau) \right\} ds$$

$$\to 0 \text{ as } j \to \infty \qquad (2.3.30)$$

Taking limits in (2.3.21) as $j \to \infty$ and making use of (2.3.18), (2.3.22), (2.3.24), (2.3.27), (2.3.28), (2.3.29) and (2.3.30), we obtain

$$x^*(t) = \begin{cases} \varphi(t) \text{ for } t \in [\alpha, t_o] \\ \varphi(t_o) + \int_{t_o}^{t} f(s, x_s^*) ds + \int_{t_o}^{t} G(s, x_s^*) du(s) \\ \quad + \int_{t_o}^{t} \left\{ \int_{t_o}^{s} H(x^*(s+t_o-\tau)) d\lambda(\tau) \right\} ds \\ \qquad\qquad \text{for } t \in (t_o, t_o+a] \end{cases}$$

x^* is thus the solution of (2.2.7) on the interval $[\alpha, t_o+a]$ with initial function φ. This completes the proof.

To show that the conditions of Theorem 3 can be satisfied without satisfying the conditions of Theorem 2, we give the following example.

EXAMPLE :

Let $n = m = 1$; $S = (-1,1)$; $t_o=0$; $\alpha = -1$, $p(t) = t-1$, $q(t) = t$. Consider the scalar equation

$$Dx = f(x_t) + G(x_t)Du + \int_0^1 H\,(x(s-\tau))\,d\tau$$

where $f(x_t) = G(x_t) = \left\{ \int_{t-1}^{t} |x(s)|\,ds \right\}^{1/2}$

and $H(\xi) = |\xi|^{1/2}$

Then since S is bounded, f and G are bounded on any finite interval $[0,T]$ and therefore H_3, H_6, H_7 and H_{10} are satisfied. H_9 is obviously true. Applying Lebesgue's dominated convergence theorem, H_2 and H_5 can easily be seen to be satisfied. But H_1, H_4 and H_8 do not hold. Consider, for example, the functions

$$x^{(n)}(t) = \begin{cases} \frac{1}{2n} \text{ if } -1 \le t \le 0 \\ \frac{1}{2n}(1-t) \text{ if } 0 < t < 1 \\ 0 \quad \text{ if } t \ge 1 \end{cases} \qquad n=1,2,\ldots\ldots$$

We have, for all values of t,

$$\| x^{(n)} \|_{[-1,t]} \le 1/n \le 1 \quad , \quad n=1,2,\ldots\ldots$$

Then if H_2 holds, there must exist a constant L such that

$$| f(x_t^{(n)}) - f(0) | \le L \| x-0 \|_{I_t} \quad \text{for all } n$$

and so,

$$|f\,(x^{(n)}_{t=1})| \leq L \,\|\, x^{(n)} \|_{I_{t=1}}$$

i.e., $1/2 \sqrt{n} \leq L \,.\, 1/n$

i.e., $\sqrt{n} \leq 2\,L$ for all values of n, which is not true. Hence H_2 does not hold.

Similarly it can be shown that H_8 is not satisfied.

Theorem 4 (Extended Existence) :

Let the hypotheses H_0, H_2, H_3, H_5, H_6, H_7, H_9 and H_{10} be satisfied with $T = \beta$. If there exists a solution $x(.;t_0,\varphi)$ of (2.2.7) on $[\alpha, \sigma)$ where σ is a point of continuity of u and if $\sigma < \beta$ and σ cannot be increased, then for any compact set $F \subset S$ there exists a sequence of numbers $t_0 < t_1 < t_2 < \ldots\ldots < t_k < \ldots\ldots \to \sigma$ such that

$$x(t_k) \in S-F \qquad \text{for } k = 1,2, \ldots\ldots\ldots$$

P R O O F : Let $x = x\,(.;\, t_0, \varphi)$ be a solution of (2.2.7) on $[\alpha, \sigma\,)$ where $\sigma \in (t_0, \beta)$ is a point of continuity of u and $\varphi \in BV\,(\,[\alpha, t_0]\;, S)$. Suppose that there exists a compact set $F \subset S$ such that $x(t) \in F$ for $t \in [\alpha, \sigma\,)$.
We have,

$$x(t) = \begin{cases} \varphi(t) \text{ for } t \in [\alpha, t_0] \\ \varphi(t_0) + \int_{t_0}^{t} f(s,x_s)ds + \int_{t_0}^{t} G(s,x_s)\; du\,(s) \\ \\ + \int_{t_0}^{t} \left\{ \int_{t_0}^{s} H(x(s+t_0-\tau))\; d\lambda(\tau) \right\} ds \text{ for } t \in (t_0, \sigma\,) \end{cases}$$

If $t_o \leq t_1 < t_2 < \sigma$, then

$$|x(t_1)-x(t_2)| \leq \int_{t_1}^{t_2} | f(s,x_s)|ds+\int_{t_1}^{t_2}|G(s,x_s)|dv_u(s)$$

$$+\int_{t_1}^{t_2}\left\{\int_{t_o}^{s} |H(x(s+t_o-\tau)) | dv_\lambda (\tau)\right\} ds$$

$$\leq \int_{t_1}^{t_2} r(s)ds + \int_{t_1}^{t_2}w(s)dv_u(s) +\int_{t_1}^{t_2}\left\{\int_{t_o}^{s} z(s+t_o-\tau)dv_\lambda(\tau)\right\} ds$$

$$= \int_{t_1}^{t_2} \gamma (s)ds + \int_{t_1}^{t_2} w(s)dv_u(s) \qquad (2.3.31)$$

where

$$\gamma (s) = r(s) + \int_{t_o}^{s} z(s+t_o-\tau)dv_\lambda (\tau)$$

Since u is continuous at the point σ, the function $\int_{t_o}^{t} w(s)dv_u(s)$ is also continuous at σ . So if $\varepsilon > 0$, we can choose t_1 so close to σ (but $t_1 < \sigma$) that

$$\int_{t_1}^{\sigma} w(s) dv_u(s) < \varepsilon /2$$

and then choose t_2 so close to t_1 that

$$\int_{t_1}^{t_2} \gamma (s) ds < \varepsilon /2$$

Thus, we obtain , from (2.3.31),

$$| x (t_1) - x(t_2) | < \varepsilon$$

for all $t_o \leq t_1 < t_2 < \sigma$ such that t_1 is sufficiently close to σ . Therefore, by Cauchy's criterion,

$$x(\sigma -) = \lim_{t\to \sigma -} x(t)$$

exists. Define

$$x (\sigma) = x (\sigma -).$$

Let

$$\varphi^* (t) = x(t) \text{ for } t \in [\alpha, \sigma].$$

Obviously, $\varphi^*(t) \in F \subset S$. Replacing t_o by σ, φ by φ^*, we can prove, as in Theorem 3, that there exists a solution $x^* = x^* (.;t_o,\varphi^*)$ of (2.2.7) on $[\alpha, \sigma+\delta]$ where $\delta > 0$, which is represented by

$$x^*(t) = \begin{cases} \varphi^*(t) \text{ for } t \in [\alpha, \sigma] \\ \varphi^*(\sigma) + \int_\sigma^t f(s,x_s)ds + \int_\sigma^t G(s,x_s)du\,(s) \\ \quad + \int_\sigma^t \left\{ \int_{t_o}^s H(x\,(s+t_o-\tau))d\lambda(\tau)ds \right. \\ \qquad\qquad \text{for } t \in (\sigma, \sigma+\delta] \end{cases}$$

It is clear that x^* is also a solution of (2.2.7) with the initial condition $x^*(t) = \varphi(t)$ for $t \in [\alpha,t_o]$, i.e. x^* is extension of x over the interval $[\alpha, \sigma+\delta]$. Thus, the value of σ can be increased. This proves the theorem.

REMARK : The set of hypotheses H_o, H_2, H_3, H_5, H_6, H_7, H_9, H_{10} is same as the one in Theorem 3. It is obvious that the conclusion of Theorem 4 also holds if these hypotheses are replaced by the hypotheses in Theorem 2 viz. H_o, H_1, H_3, H_4, H_6, H_8 and H_{10}.

I. I. T. KANPUR
CENTRAL LIBRARY
Acc. No A. 19835

CHAPTER 3

FUNCTIONAL INTEGRO - DIFFERENTIAL EQUATIONS CONTAINING MEASURES (SPECIAL CASE)

3.1 INTRODUCTION

This chapter is devoted to the study of equation (2.2.7) in the special case when the functional G is independent of x. In this case it has been possible to give constructive proof of solution by using Picard's approximation. In addition to this existence theorem, dependence of the solutions on the initial function and on the right hand side of the equation is also considered. The failure to extend these results for the general case when G depends on both t and x, is due to inavailability of integral inequalities (of the type of Gronwall's lemma) for Stieltjes integrals.

3.2 EXISTENCE AND UNIQUENESS OF SOLUTION

Consider the measure integro - differential equation

$$Dx = f(t,x_t) + G(t)Du + \int_{t_0}^{t} H(x(t+t_0-\tau))d\lambda(\tau), \quad t>t_0 \tag{3.2.1}$$

which is a special case of (2.2.7) with $G(t,x_t) \equiv G(t)$ depending only on t. By Theorem 1, a function x is a solution of (3.2.1) with initial function $\varphi \in BV([\alpha,t_o],S)$ iff it is a solution of the integral equation

$$x(t) = \begin{cases} \varphi(t) \text{ for } t \in [\alpha,t_o] \\ \varphi(t_o)+\int_{t_o}^{t} f(s,x_s)ds+\int_{t_o}^{t} G(s)\,du(s) \\ +\int_{t_o}^{t}\left\{\int_{t_o}^{s} H(x(s+t_o-\tau))d\lambda(\tau)\right\} ds,\ t>t_o \end{cases} \quad (3.2.2)$$

We shall now give a constructive proof of the following existence theorem using Picard's successive approximation procedure.

<u>Theorem 5</u> (Local Existence and Uniqueness) :

Let the hypotheses H_o, H_1, H_3, H_8 and H_{10} (of section 2.3) be satisfied, and let $\varphi \in BV([\alpha,t_o],S)$. Then there exists a unique solution x of (3.2.1) on an interval $[\alpha,t_o+a]$, $a>0$, with initial function φ.

P R O O F : Define Q_t as in the proof of Theorem 2, and choose a , $0< a < T-t_o$, such that

$$\int_{t_o}^{t_o+a} r(s)ds + \int_{t_o}^{t_o+a} |G(s)|\,dv_u(s) \quad (3.2.3)$$

$$+v\ (\lambda, [t_o,t_o+a])\int_{t_o}^{t_o+a} z(s)ds \le b$$

Let A be the transformation defined by

$$(Ax)(t) = \begin{cases} \varphi(t) \text{ for } t \in [\alpha,t_o] \\ \varphi(t_o)+\int_{t_o}^{t} f(s,x_s)ds+\int_{t_o}^{t} G(s)du(s) \\ +\int_{t_o}^{t}\left\{\int_{t_o}^{s} H(x(s+t_o-\tau))d\lambda(\tau)\right\} ds \\ \quad \text{for } t \in (t_o,t_o+a] \end{cases} \quad (3.2.4)$$

Then as in Theorem 2, A maps Q_{t_o+a} into itself.

Now define

$$x^{(0)}(t) = \begin{cases} \varphi(t) & \text{for } t \in [\alpha, t_o] \\ \varphi(t_o) + \int_{t_o}^{t} G(s)du(s) & \text{for } t \in (t_o, t_o+a] \end{cases} \tag{3.2.5}$$

and

$$x^{(k)}(t) = \begin{cases} \varphi(t) \text{ for } t \in [\alpha, t_o] \\ \varphi(t_o) + \int_{t_o}^{t} f(s, x_s^{(k-1)})ds + \int_{t_o}^{t} G(s)du(s) \\ \quad + \int_{t_o}^{t} \left\{ \int_{t_o}^{s} H(x^{(k-1)}(s+t_o-\tau))d\lambda(\tau) \right\} ds \\ \qquad \text{for } t \in (t_o, t_o+a] \end{cases} \tag{3.2.6}$$

$k = 1, 2, \ldots\ldots$

Clearly $x^{(o)} \in Q_{t_o+a}$, and since

$$A\, x^{(k-1)} = x^{(k)} \tag{3.2.7}$$

and A maps Q_{t_o+a} into itself, we obtain

$$x^{(k)} \in Q_{t_o+a}, \quad k = 0, 1, 2, \ldots\ldots \tag{3.2.8}$$

and so

$$v(x^{(k)}, [t_o, t_o+a]) \leq b, \quad k=0,1,2\ldots \tag{3.2.9}$$

Therefore,

$$\| x^{(k)} \|_{[\alpha, t_o+a]} \leq \|\varphi\|_{[\alpha, t_o]} + b$$
$$= c_1, \text{ say} \tag{3.2.10}$$

Also,

$$|x^{(k)}(s)| \leq \| x^{(k)} \|_{[\alpha, t_o+a]} \leq c_1 \tag{3.2.11}$$

for each $s \in [\alpha, t_o+a]$. Therefore, by H_1 and H_8, there exist constants $c_2 = c_2(c_1)$ and $c_3 = c_3(c_1)$ such that

$$\begin{aligned} &| f(s,x_s^{(k)}) - f(s,x_s^{(k-1)})| \\ &\leq c_2 \| x^{(k)} - x^{(k-1)} \|_{I_s} \\ &\leq c_2 \| x^{(k)} - x^{(k-1)} \|_{[\alpha,s]} , \end{aligned} \tag{3.2.12}$$

and

$$\begin{aligned} &|H(x^{(k)}(s+t_o-\tau)) - H(x^{(k-1)}(s+t_o-\tau))| \\ &\leq c_3 \, |x^{(k)}(s+t_o-\tau) - x^{(k-1)}(s+t_o-\tau)| \\ &\leq c_3 \| x^{(k)} - x^{(k-1)} \|_{[\alpha,s]} \end{aligned} \tag{3.2.13}$$

for all $s \in [t_o, t_o+a]$ and $k = 1, 2, \ldots$.

Now, for each $t \in [t_o, t_o+a]$, we have

$$\begin{aligned} &\| x^{(k+1)} - x^{(k)} \|_{[\alpha,t]} \\ &\leq \int_{t_o}^{t} |f(s,x_s^{(k)}) - f(s,x_s^{(k-1)})| ds \\ &+ \int_{t_o}^{t} \Big\{ \int_{t_o}^{s} |H(x^{(k)}(s+t_o-\tau)) \\ &\qquad -H(x^{(k-1)}(s+t_o-\tau)| dv_\lambda(\tau) \Big\} ds \\ &\leq c_2 \int_{t_o}^{t} \| x^{(k)} - x^{(k-1)} \|_{[\alpha,s]} ds \\ &+ c_3 \int_{t_o}^{t} \Big\{ \int_{t_o}^{s} \| x^{(k)} - x^{(k-1)} \|_{[\alpha,s]} \, dv_\lambda(\tau) \Big\} ds, \end{aligned}$$

by (3.2.12) and (3.2.13)

$$\leq L \int_{t_o}^{t} \| x^{(k)} - x^{(k-1)} \|_{[\alpha,s]} ds, \tag{3.2.14}$$

where

$$L = c_2 + c_3\, v\,(\lambda\,,\,[t_o,\, t_o + a]\,). \tag{3.2.15}$$

Furthermore, we have for each $t \in [t_o, t_o+a]$,

$$\| x^{(1)} - x^{(0)} \|_{[\gamma,t]} \leq \| x^{(1)} \|_{[\gamma,t]} + \| x^{(0)} \|_{[\gamma,t]}$$

$$\leq 2\, c_1, \text{ by } (3.2.10)$$

$$= c, \text{ say.} \tag{3.2.16}$$

Also, by (3.2.14) and (3.2.16),

$$\| x^{(2)} - x^{(1)} \|_{[\gamma,t]} \leq L \int_{t_o}^{t} c\, ds = c\, L(t-t_o) \tag{3.2.17}$$

By (3.2.14) and (3.2.17), we obtain

$$\| x^{(k+1)} - x^{(k)} \|_{[\gamma,t]} \leq c\, L^k (t-t_o)^k / k!$$

$$k = 1, 2, \ldots. \tag{3.2.18}$$

for all $t \in [t_o, t_o+a]$. Hence $x^{(k)}$ is a Cauchy sequence in Q_{t_o+a} which is a complete metric space. Thus $x^{(k)}$ converges pointwise to a limit function x in Q_{t_o+a} . Moreover, due to (3.2.7), we have

$$\| Ax - x \|_{[\gamma, t_o+a]} \leq \| Ax - Ax^{(k)} \|_{[\gamma, t_o+a]}$$

$$+ \| x^{(k+1)} - x \|_{[\gamma, t_o+a]}$$

and we obtain, as in (3.2.14),

$$\| Ax - Ax^{(k)} \|_{[\gamma, t_o+a]} \leq L \int_{t_o}^{t_o+a} \| x - x^{(k)} \|_{[\alpha, s]}\, ds$$

$$\leq aL \| x - x^{(k)} \|_{[\gamma, t_o+a]}$$

Thus

$$\| Ax - x \|_{[\gamma, t_o+a]} \leq aL \| x - x^{(k)} \|_{[\gamma, t_o+a]} + \| x^{(k+1)} - x \|_{[\gamma, t_o+a]}$$

which converges to zero as $k \to \infty$.

Therefore

$$Ax = x$$

i.e.,

$$x(t) = \begin{cases} \varphi(t) \text{ for } t \in [\gamma, t_o] \\ \varphi(t_o) + \int_{t_o}^{t} f(s, x_s)ds + \int_{t_o}^{t} G(s)du(s) \\ \quad + \int_{t_o}^{t} \left\{ \int_{t_o}^{s} H(x(s+t_o-\tau))\, d\lambda(\tau) \right\} ds \\ \qquad \text{for } t \in (t_o, t_o+a] \end{cases}$$

Thus, x is a solution of (3.2.1) on $[\gamma, t_o+a]$ with initial function φ.

To prove uniqueness, let y be any solution of (3.2.1) on $[\gamma, t_o+a]$ with initial function φ. Then

$$y(t) = \begin{cases} \varphi(t) \text{ for } t \in [\gamma, t_o] \\ \varphi(t_o) + \int_{t_o}^{t} f(s, y_s)ds + \int_{t_o}^{t} G(s)du(s) \\ \quad + \int_{t_o}^{t} \left\{ \int_{t_o}^{s} H(y(s+t_o-\tau))\, d\lambda(\tau) \right\} ds \\ \qquad \text{for } t \in (t_o, t_o+a] \end{cases}$$

Using (3.2.6), it can be shown (Cf.(3.2.18)) that

$$\| x^{(k+1)} - y \|_{[\gamma,t]} \leq c_o \frac{L^k (t-t_o)^k}{k!}, \; k=1,2,.. \quad (3.2.19)$$

where

$$\| x^{(1)} - y \|_{[\gamma,t]} \leq c_o .$$

If $k \to \infty$ in (3.2.19), it follows that

$$\| x - y \|_{[\gamma,t]} = 0$$

for every $t \in [t_o, t_o+a]$, i.e. $x \equiv y$. This completes the proof.

REMARK : It can be shown that in Theorem 3 the hypotheses H_5, H_6, H_7 concerning G can be dropped in case G depends only on t. In this case one can choose 'a' so as to satisfy (3.2.3) and then define the iterants by

$$x^{(k)}(t) = \begin{cases} \bar{\varphi}(t) \text{ for } t \in [\alpha, t_o+a/k] \\ \varphi(t_o)+\int_{t_o}^{t-a/k} f(s,x_s^{(k)})ds+\int_{t_o}^{t} G(s)du(s) \\ \quad +\int_{t_o}^{t-a/k} \left\{\int_{t_o}^{s} H(x^{(k)}(s+t_o-\tau))d\lambda(\tau)\right\}ds \\ \qquad\qquad \text{for } t \in (t_o+a/k, t_o+a] \end{cases}$$

$k = 1, 2, \ldots.$

The rest of the proof is same as that of Theorem 3.

3.3 DEPENDENCE OF SOLUTIONS ON THE INITIAL FUNCTION AND THE RIGHT - HAND SIDE OF THE EQUATION

Consider the two measure integro-differential equations

$$Dx = f_1(t,x_t) + G_1(t)Du+\int_{t_o}^{t} H_1(x(t+t_o-\tau))d\lambda(\tau) \quad (3.3.1)$$

$$Dy = f_2(t,y_t)+G_2(t)Du+\int_{t_o}^{t} H_2(y(t+t_o-\tau))d\lambda(\tau) \quad (3.3.2)$$

Assume that f_1, f_2, H_1, H_2 are locally Lipschitzian in x, and that

$$|f_1(t,x_t)-f_2(t,x_t)| \leq \varepsilon_1 , \ (t,x_t) \in E \quad (3.3.3)$$

$$|G_1(t)-G_2(t)| \leq \varepsilon_2 , \ t \in [t_o,\beta] \quad (3.3.4)$$

$$| H_1 (\xi) - H_2(\xi)| \leq \varepsilon_3 , \ \xi \in S. \quad (3.3.5)$$

Assume further that G_1 and G_2 are measurable with respect to LS-measure du, and that for each $x \in BV([t_o, \beta])$, $f_1(t,x_t)$, $f_2(t,x_t)$ are Lebesgue measurable and $H_1(x(t))$, $H_2(x(t))$ are measurable with respect to LS-measure $d\lambda$. Let $x = x(.;t_o,\varphi)$ and $y = y(.;t_o,\psi)$ be solutions of (3.3.1) and (3.3.2) respectively on $[\gamma, t_o+a]$ where $\varphi, \psi \in BV([\gamma, t_o], S)$. Then

$$x(t)-y(t) = \begin{cases} \varphi(t)-\psi(t) \text{ for } t \in [\gamma, t_o] \\ \varphi(t_o) - \psi(t_o) + \int_{t_o}^{t} [f_1(s,x_s)-f_2(s,y_s)]ds \\ \quad + \int_{t_o}^{t} [G_1(s)-G_2(s)] \, du(s) \\ \quad + \int_{t_o}^{t} \left\{ \int_{t_o}^{s} [H_1(x(s+t_o-\tau)) \right. \\ \qquad \left. -H_2(y(s+t_o-\tau))] \, d\lambda(\tau) \right\} ds \\ \qquad \text{for } t \in (t_o, t_o+a] \end{cases}$$

from which we obtain

$$\|x-y\|_{[\gamma,t]} \leq \begin{cases} \|\varphi-\psi\|_{[\gamma,t_o]} \text{ for } t \in [\gamma, t_o] \\ \|\varphi-\psi\|_{[\gamma,t_o]} \\ +\int_{t_o}^{t} |f_1(s,x_s)-f_2(s,y_s)|ds \\ +\int_{t_o}^{t} |G_1(s)-G_2(s)| dv_u(s) \\ +\int_{t_o}^{t} \left\{ \int_{t_o}^{s} |H_1(x(s+t_o-\tau)) \right. \\ \qquad \left. -H_2(y(s+t_o-\tau))| dv_\lambda(\tau) \right\} ds \\ \qquad \text{for } t \in (t_o, t_o+a] \end{cases} \tag{3.3.6}$$

But

$$\int_{t_o}^{t} |f_1(s,x_s)-f_2(s,y_s)|ds$$

$$\leq \int_{t_o}^{t} |f_1(s,x_s)-f_1(s,y_s)|ds$$

$$+ \int_{t_o}^{t} |f_1(s,y_s)-f_2(s,y_s)|ds$$

$$\leq c_1 \int_{t_o}^{t} \| x-y\|_{I_s} ds + \varepsilon_1(t-t_o) \tag{3.3.7}$$

$$\leq \varepsilon_1 a + c_1 \int_{t_o}^{t} \| x-y \|_{[\gamma,s]} ds ,$$

where c_1 is Lipschitz constant for f_1 on $[t_o,t_o+a]$

Also,

$$\int_{t_o}^{s} |H_1(x(s+t_o-\tau))-H_2(y(s+t_o-\tau))| dv_\lambda(\tau)$$

$$\leq \int_{t_o}^{s} |H_1(x(s+t_o-\tau))-H_1(y(s+t_o-\tau))| dv_\lambda(\tau)$$

$$+ \int_{t_o}^{s} H_1(y(s+t_o-\tau))-H_2(y(s+t_o-\tau))| dv_\lambda(\tau)$$

$$\leq c_2 \int_{t_o}^{s} |x(s+t_o-\tau)-y(s+t_o-\tau)| dv_\lambda(\tau)$$

$$+\varepsilon_3 v(\lambda, [t_o,s]),$$

c_2 being the Lipschitz constant for H_1 on $[t_o,t_o+a]$

$$\leq \varepsilon_3 v(\lambda, [t_o,t_o+a])+c_2 \int_{t_o}^{s} \| x-y\|_{[\gamma,s]} dv_\lambda(\tau)$$

$$\leq \varepsilon_3 v(\lambda, [t_o,t_o+a]) \tag{3.3.8}$$

$$+c_2 v(\lambda, [t_o,t_o+a])\| x-y\|_{[\gamma,s]}$$

From (3.3.6), (3.3.7), (3.3.8) and (3.3.4) we obtain for $t \in [t_o, t_o+a]$,

$$\| x-y \|_{[\tau,t]} \leq \|\varphi - \psi\|_{[\tau,t_o]} + \varepsilon_1 a + c_1 \int_{t_o}^{t} \| x-y \|_{[\tau,s]} ds$$

$$+\varepsilon_2 v(u, [t_o,t_o+a]) + \varepsilon_3 a v(\lambda, [t_o,t_o+a])$$

$$+ c_2 v(\lambda, [t_o,t_o+a]) \int_{t_o}^{t} \|x-y\|_{[\tau,s]} ds$$

$$= c_o + c\int_{t_o}^{t} \|x-y\|_{[\tau,s]} ds \qquad (3.3.9)$$

where

$$c_o = \|\varphi - \psi\|_{[\tau,t_o]} + \varepsilon_1 a + \varepsilon_2 v(u, [t_o,t_o+a]) + \varepsilon_3 a v(\lambda, [t_o,t_o+a]) \qquad (3.3.10)$$

and

$$c = c_1 + c_2 v(\lambda, [t_o,t_o+a]). \qquad (3.3.11)$$

Applying Gronwall's inequality to (3.3.9), we obtain

$$\| x-y \|_{[\tau,t]} \leq c_o e^{c(t-t_o)}, \quad t \in [t_o,t_o+a] \qquad (3.3.12)$$

If we take $f_1 = f_2 = f$, $G_1 = G_2 = G$, $H_1=H_2=H$,

then we have the following :

Theorem 6 (Continuous Dependence on Initial Function):

Let the hypotheses H_0, H_1, H_3, H_8 and H_{10} be satisfied. Consider the solutions $x = x(.;t_0,\varphi)$ and $y = y(.;t_0,\psi)$ of (3.2.1) on the interval $[\gamma, t_0+a]$ (their existence and uniqueness are known by Theorem 5) satisfying the initial conditions $x(t) = \varphi(t)$, $y(t) = \psi(t)$ for $t \in [\gamma, t_0]$. Let $\varepsilon > 0$ be given. Then there exists a $\delta > 0$ such that

$$\| x-y \|_{[\gamma, t]} < \varepsilon$$

whenever

$$\| \varphi - \psi \|_{[\gamma, t_0]} < \delta \ ;$$

i.e. the solution $x = x(.;t_0, \varphi)$ of (3.2.1) depends continuously upon the initial function φ.

For, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$ and from (3.3.10) and (3.3.12) we have

$$\| x-y \|_{[\gamma, t]} \leq \| \varphi - \psi \|_{[\gamma, t_0]} e^{ca}$$

and then we can take

$$\delta = \varepsilon / e^{ca} .$$

Next, if we take $\varphi = \psi$ then we obtain from (3.3.10) and (3.3.12) :

$$\| x-y \|_{[\tau,t]} \leq [\varepsilon_1 a + \varepsilon_2 v(u, [t_o,t_o+a])$$
$$+ \varepsilon_3 a v(\lambda, [t_o,t_o+a])] e^{ca} \tag{3.3.13}$$

Clearly the right-hand side of (3.3.13) can be made arbitrarily small by choosing ε_1 , ε_2 and ε_3 sufficiently small. We have thus the following :

<u>Theorem 7</u> (Continuous Dependence on r.h.s.) :

Let H_o, H_1, H_3, H_8 and H_{10} be satisfied. Then the solution $x = x(.; t_o, \varphi)$ of (3.2.1) on the interval $[\tau, t_o+a]$ and satisfying the initial condition $x(t)=\varphi(t)$ for $t \in [\tau, t_o]$ depends continuously on the right-hand side of (3.2.1).

CHAPTER 4

ORDINARY DIFFERENTIAL EQUATIONS CONTAINING MEASURES

4.1 INTRODUCTION

In this chapter we shall consider the following measure differential equation

$$Dx = f(t,x) + G(t,x)\ Du \tag{4.1.1}$$

where f and G are defined on $R^+ \times R^n$ (R^+ is positive real line) with values in R^n and the set $\mathfrak{M}$ of all n x m matrices respectively, and u is a right continuous function $\in BV(R^+,R^m)$. Equation (4.1.1) is a special case of (2.2.7) when $p(t) = q(t) = t$ and $H \equiv 0$. Following Definition 1 in Chapter 2, we shall call function $x = x(\cdot;t_0,x_0)$ a solution of (4.1.1) on the interval I through (t_0,x_0) if x is right continuous function $\in BV(I,S)$ (S is a domain in R^n), $x(t_0)= x_0$ and the distributional derivative of x on (t_0,T) for any arbitrary $T \in I$ satisfies (4.1.1). According to Theorem 1 , $x(\cdot)$ is a solution of (4.1.1) on I through (t_0,x_0) if and only if it satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s,x(s))ds+\int_{t_0}^{t} G(s,x(s))du(s) \tag{4.1.2}$$

for $t \in I$. It will be shown in section 4.2 that in the local existence theorem for (4.1.1), corresponding to Theorem 3, the hypothesis H_5 can be replaced by the continuity of G in t.

The primary aim of considering (4.1.1) is the following. The equation (4.1.1) may be regarded as perturbed system of the ordinary differential equation

$$\frac{dx}{dt} = f(t,x) \tag{4.1.3}$$

where the perturbation Du is impulsive and has the effect of suddenly changing the state of the system. A natural question arises under what conditions the stability properties of (4.1.3) are shared by the solutions of (4.1.1). It seems very difficult to get a satisfactory answer to this question. It may be so because differential and integral inequalities play a very important role in the stability theory ; but when we consider the stability of solutions of (4.1.1), the fact that its solutions are discontinuous renders many of differential inequalities unapplicable while the integral inequalities are not available for the Stieltjes integrals. However, section 4.3 of the present chapter deals with a stability theorem under conditions which may be regarded to be restrictive. Such conditions are natural to be expected, since otherwise the discontinuities of u(t) may give considerable impulsive changes in the state variables to make the system unstable.

4.2 EXISTENCE OF SOLUTION

Define

$$S_b(x_o) = \{ x \in R^n : |x-x_o| < b \}$$

$$E = \{ (t,x) : t \in [t_o,t_1],\ x \in S_b(x_o) \}$$

<u>Theorem 8</u> :

Suppose that f and G are defined on E and that the following conditions are satisfied :

(i) $f(t,x)$ is measurable in t for each x ;

(ii) $f(t,x)$ is continuous in x for each $t \in [t_o,t_1]$;

(iii) there exists a Lebesgue integrable function r such that

$$|f(t,x)| \leq r(t) \quad ,(t,x) \in E$$

(iv) $G(t,x(t))$ is du - integrable for each $x \in BV([t_o,t_1], S_b(x_o))$;

(v) $G(t,x)$ is continuous in x for each t ;

(vi) there exists a dv_u-integrable function w such that

$$|G(t,x)| \leq w(t), \quad (t,x) \in E.$$

Then there exists a solution $x(.)$ of (4.1.1) on some interval $[t_o,t_o+a]$ satisfying the initial condition $x(t_o) = x_o$.

PROOF : Choose a $(0 < 2a < t_1 - t_o)$ such that

$$\int_{t_o}^{t_o+2a} r(s)ds + \int_{t_o}^{t_o+2a} w(s)dv_u(s) \leq c \qquad (4.2.1)$$

where $0 < c < b$. Since $\int_{t_o}^{t} r(s)ds$ is continuous function of t and $\int_{t_o}^{t} w(s)dv_u(s)$ is right continuous function of t, it is possible to choose such an a.

Now consider the following integral equation

$$x^{(k)}(t) = \begin{cases} x_o \text{ for } t \in [t_o - 2a/k, t_o] \\ x_o + \int_{t_o}^{t} f(s, x^{(k)}(s-2a/k))ds \\ +\int_{t_o}^{t} G(s, x^{(k)}(s-2a/k))du(s) \\ \quad \text{for } t \in (t_o, t_o+2a] \end{cases} \qquad (4.2.2)$$

For any k, the first expression in (4.2.2) defines $x^{(k)}$ on $[t_o-2a/k, t_o]$ where $x^{(k)}(t) = x_o$, and since $(t, x_o) \in E$ for $t \in [t_o, t_o+2a/k]$ the second expression of (4.2.2) defines $x^{(k)}$ as a function of bounded variation on the interval $(t_o, t_o+2a/k]$. Let us assume that $x^{(k)}$ is defined on $[t_o-2a/k, t_o+2aj/k]$, $1 \leq j < k$. Then, we have for $t \in [t_o-2a/k, t_o+2aj/k]$,

$$\begin{aligned} |x^{(k)}(t) - x_o| &\leq v(x^{(k)}(.) - x_o, [t_o-2a/k, t_o+2aj/k]) \\ &\leq \int_{t_o}^{t_o+2aj/k} |f(s, x^{(k)}(s-2a/k)| ds \\ &\quad +\int_{t_o}^{t_o+2aj/k} |G(s, x^{(k)}(s-2a/k)| dv_u(s) \qquad (4.2.3) \\ &\leq c < b \text{ , by (4.2.1)} \end{aligned}$$

and therefore the second expression of (4.2.2) defines $x^{(k)}$ on $(t_o+2aj/k,\ t_o+2a(j+1)/k]$. Thus $x^{(k)}$ is defined on $[t_o-2a/k,\ t_o+2a]$ and it can be seen, as in (4.2.3), that

$$|x^{(k)}(t) - x_o| \leq v(x^{(k)}(.)-x_o,\ [t_o-2a/k, t_o+2a]) < b \tag{4.2.4}$$

Define

$$\tilde{x}^{(k)}(t+2a/k) = x^{(k)}(t),\ t_o-2a/k \leq t \leq t_o+2(1-1/k)a,\quad k \geq 2 \tag{4.2.5}$$

From the right continuity of $x^{(k)}$, it follows that $\tilde{x}^{(k)}$ is also right continuous. Also, by (4.2.4), $\tilde{x}^{(k)}$ are of uniform bounded variation on $[t_o, t_o+a]$. By Helly's selection principle, there exists a subsequence $x^{(k_j)}$ and a function x^* such that

$$\lim_{j\to\infty} \tilde{x}^{(k_j)}(t) = x^*(t) \quad , t \in [t_o, t_o+a] \tag{4.2.6}$$

Therefore, by virtue of conditions (ii) and (iv), and (4.2.6),

$$\lim_{j\to\infty} f(t,\tilde{x}^{(k_j)}(t)) = f(t,x^*(t)) \tag{4.2.7}$$

$$\lim_{j\to\infty} G(t,\tilde{x}^{(k_j)}(t)) = G(t,x^*(t)) \tag{4.2.8}$$

By Lebesgue's dominated convergence theorem, we have

$$\lim_{j\to\infty} \int_{t_o}^{t} f(s,\tilde{x}^{(k_j)}(s))ds = \int_{t_o}^{t} f(s,x^*(s))ds$$

Also,

$$\lim_{j\to\infty} \int_{t_o}^{t} G(s,\tilde{x}^{(k_j)}(s))du(s)$$

$$= \lim_{j\to\infty}\Big[\int_{t_o}^{t} G(s,\tilde{x}^{(k_j)}(s))du^{+}(s)$$

$$- \int_{t_o}^{t} G(s,\tilde{x}^{(k_j)}(s))du^{-}(s)\Big] ,$$

where du^+ and du^- are positive and negative variations of LS-measure du.

$$= \int_{t_o}^{t} G(s,x^{*}(s))du^{+}(s) - \int_{t_o}^{t} G(s,x^{*}(s))du^{-}(s) ,$$

by (4.2.8) using Lebesgue's dominated convergence theorem.

$$= \int_{t_o}^{t} G(s,x^{*}(s))du(s).$$

From (4.2.2), we write

$$\tilde{x}^{(k_j)}(t+2a/k_j) = \begin{cases} x_o \text{ for } t \in [t_o-2a/k_j,t_o] \\ x_o+ \int_{t_o}^{t} f(s,\tilde{x}^{(k_j)}(s))ds \\ \quad + \int_{t_o}^{t} G(s,\tilde{x}^{(k_j)}(s))du(s) \\ \qquad \text{for } t \in (t_o,t_o+a] \end{cases}$$

Taking limit as $j \to \infty$, we have

$$x^{*}(t) = x_o+\int_{t_o}^{t} f(s,x^{*}(s))ds+\int_{t_o}^{t} G(s,x^{*}(s))du(s)$$

for $t \in [t_o,t_o+a]$. Hence x^* is a solution of (4.1.1) on $[t_o, t_o+ a]$ through (t_o, x_o).

4.3 STABILITY

We begin by recalling certain definitions.

Let $x(\cdot;t_o,x_o)$ be any solution of the differential equation

$$\frac{dx}{dt} = f(t,x), \quad x(t_o) = x_o, \quad t_o \geq 0 \qquad (4.3.1)$$

where f is defined and continuous on $R^+ \times S_\rho$, S_ρ being the set

$$S_\rho = \left\{ x \in R^n : |x| < \rho \right\} .$$

We assume that $f(t,0) = 0$, $t \in R^+$, so that $x = 0$ is a (trivial) solution of (4.3.1) through $(t_o, 0)$.

DEFINITION [19, (S_3), p. 136]: The trivial solution $x = 0$ of (4.3.1) is quasi - equi asymptotically stable if, for each $\varepsilon > 0$, $t_o \in R^+$, there exist positive numbers $\delta_o = \delta_o(t_o)$ and $T = T(t_o,\varepsilon)$ such that, for $t \geq t_o+T$ and $|x_o| < \delta_o$,

$$|x(t;t_o,x_o)| < \varepsilon .$$

DEFINITION [19, p. 158]: The trivial solution of (4.3.1) is said to be exponentially asymptotically stable if there exist a $c > 0$ and a $K > 0$ such that

$$|x(t;t_o,x_o)| \leq K e^{-c(t-t_o)} |x_o| .$$

A scalar function $V(t,x)$ defined on $R^+ \times S$ is called a Lyapunov function if it is continuous in (t,x) and locally Lipschitz in x. If $V(t,x)$ is a Lyapunov function then we define

$$V'_{(4.1.3)}(t,x) = \overline{\lim_{h\to 0+}} \frac{1}{h} \left\{ V(t+h, x+hf(t,x)) - V(t,x) \right\}$$

$$= \overline{\lim_{h\to 0+}} \frac{1}{h} \left\{ V(t+h, x(t+h)) - V(t,x) \right\}$$

where $x(.)$ is the solution of (4.1.3) through (t,x).

We shall now consider the perturbed system

$$Dx = f(t,x) + G(t,x)\, Du \qquad (4.3.2)$$

The following theorem will be proved.

<u>Theorem 9</u> :

Let the trivial solution of (4.3.1) be exponentially asymptotically stable, i.e. there exist a $c > 0$ and a $K > 0$ such that

$$|x(t;t_o,x_o)| \leq K\, e^{-c(t-t_o)} |x_o|.$$

Suppose that

(1) $f(t,x)$ satisfies Lipschitz condition in x for a constant $M = M(\rho) > 0$;

(2) $|G(t,x)| \leq g(t)\, |x|$;

(3) $\int_0^\infty \omega(t)dt < \infty$, where

$$\omega(t) = \overline{\lim_{h\to 0+}} \frac{1}{h} \int_t^{t+h} g(s)dv_u(s)$$

is the upper right Dini derivative of the indefinite integral $\int_0^t g(s)dv_u(s)$;

(4) the discontinuities $t_1 < t_2 < \ldots\ldots < t_k < \ldots$ of u are isolated, and are such that

$$|u(t_k) - u(t_k-)| \leq \varepsilon\, e^{-c(t_k-t_o)}.$$

Then if $\sum_{k=1}^{\infty} g(t_k)$ converges and ε is sufficiently small, the trivial solution $x = 0$ of (4.3.2) is quasi - equi asymptotically stable.

PROOF : By [19, Th. 3.6.2 and Cor. 3.6.1] or [38, Cor. of Th. 19.2], there exists a Lyapunov function $V(t,x)$ defined on $R^+ \times S_\rho$ with the following properties

(i) $|V(t,x)-V(t,y)| \leq L\,|x-y|$

(ii) $|x| \leq V(t,x) \leq K|x|$

(iii) $V'_{(4.3.1)}(t,x) \leq -\beta c\, V(t,x)$, where $0 < \beta < 1$.

Consider a function $W(t,x)$ defined on $R^+ \times S_\rho$ by

$$W(t,x) = V(t,x)\exp\left\{-L\int_0^t \omega(s)ds\right\} \qquad (4.3.3)$$

Let $x(.;\, t,x)$ and $\overset{*}{x}(.;\, t,x)$ be solutions through (t,x) of (4.3.1) and (4.3.2) respectively. We have,

$$W'_{(4.3.2)}(t,x) = \overline{\lim_{h\to 0+}}\ \frac{1}{h}\left\{W(t+h,x^*(t+h;t,x))-W(t,x)\right\}$$

$$= \overline{\lim_{h\to 0+}}\ \frac{1}{h}\left\{W(t+h,x^*(t+h;t,x)) - W(t+h,x(t+h;t,x))\right\}$$

$$+ \overline{\lim_{h\to 0+}}\ \frac{1}{h}\left\{W(t+h,x(t+h;t,x)) - W(t,x)\right\}$$

$$=\overline{\lim_{h\to 0+}}\ \frac{1}{h}\, e^{-L\int_0^{t+h}\omega(s)ds}\Big\{V(t+h,x^*(t+h;t,x))-V(t+h,x(t+h;t,x))\Big\}$$

$$+W'_{(4.3.1)}(t,x)$$

$$=\overline{\lim_{h\to 0+}}\ \frac{1}{h}\, e^{-L\int_0^{t+h}\omega(s)ds}\Big\{V\Big[t+h,x+\int_t^{t+h} f(s,x^*(s;t,x))ds$$

$$+\int_t^{t+h} G(s,x^*(s;t,x))du(s)\Big]$$

$$-V\Big[t+h,x+\int_t^{t+h} f(s,x(s;t,x))ds\Big]\Big\}$$

$$+e^{-L\int_0^t\omega(s)ds}\Big\{V'_{(4.3.1)}(t,x)-L\omega(t)\,V(t,x)\Big\}$$

$$\leq \overline{\lim_{h\to 0+}}\ \frac{1}{h}\, e^{-L\int_0^{t+h}\omega(s)ds}\Big\{L\int_t^{t+h}|f(s,x^*(s;t,x))$$

$$-f(s,x(s;t,x))|ds$$

$$+L\int_t^{t+h}|G(s,x^*(s;t,x))|dv_u(s)\Big\}$$

$$+e^{-L\int_0^t\omega(s)ds}\Big\{-\beta cV(t,x)-L\omega(t)V(t,x)\Big\}$$

$$\leq \overline{\lim_{h\to 0+}}\ \frac{1}{h}\, e^{-L\int_0^{t+h}\omega(s)ds}\Big\{LM\sup_{t\leq s\leq t+h}|x^*(s;t,x)-x(s;t,x)|h$$

$$+L\sup_{t\leq s\leq t+h}|x^*(s;t,x)|\int_t^{t+h}g(s)dv_u(s)\Big\}$$

$$+e^{-L\int_0^t\omega(s)ds}\Big\{-\beta cV(t,x)-L\omega(t)V(t,x)\Big\}$$

$$= e^{-L\int_0^t \omega(s)ds} \left\{ L\omega(t)|x| - \beta c V(t,x) - L\omega(t)V(t,x) \right\}$$

$$\leq e^{-L\int_0^t \omega(s)ds} \left\{ L\omega(t)V(t,x) - \beta c V(t,x) - L\omega(t)V(t,x) \right\}$$

$$= -\beta c\ W(t,x) \tag{4.3.4}$$

Since $x^*(.;t_o,x_o)$ is continuous on $[t_{k-1},t_k)$, $k=1,2,\ldots$, we have, by [38,Th.4.1],

$$W(t,x^*(t;t_o,x_o)) \leq W(t_{k-1},x^*(t_{k-1};t_o,x_o))\ e^{-\beta c(t-t_{k-1})} \quad \text{for } t \in [t_{k-1},t_k) \tag{4.3.5}$$

We have ,

$$x^*(t;t_o,x_o) - x^*(t-;t_o,x_o) = \lim_{h\to 0+} \left\{ \int_{t-h}^{t} f(s,x^*(s;t_o,x_o))ds + \int_{t-h}^{t} G(s,x^*(s;t_o,x_o))du(s) \right\} \tag{4.3.6}$$

The first limit on the right is zero ; and we shall prove that

$$\lim_{h\to 0+} \left| \int_{t-h}^{t} G(s,x^*(s;t_o,x_o))du(s) \right| = \left| G(t,x^*(t;t_o,x_o))(u(t)-u(t-)) \right| \tag{4.3.7}$$

Consider the positive measure μ defined by

Let $h_1 \geq h_2 \geq h_3 \geq \ldots > 0$ and let $A_k = [t-h_k, t]$ and $h_k \to 0$ as $k \to \infty$. Then $A_1 \supset A_2 \supset A_3 \supset \ldots$ and $\bigcap_{k=1}^{\infty} A_k = A_o$ where $A_o = \{t\}$. Therefore, $\mu(A_k) \to \mu(A_o)$ (Rudin [33], Th.1.19(e)). But $\mu(A_o) = |G(t,x^*(t;t_o,x_o))(u(t)-u(t-))|$, by Munroe [23, Ex.s,p.199]. Therefore (4.3.7) is established ; and we have, from (4.3.6),

$$|x^*(t;t_o,x_o)-x^*(t-;t_o,x_o)|$$
$$=|G(t,x^*(t;t_o,x_o))(u(t)-u(t-))| \qquad (4.3.8)$$

Now,

$$|W(t,x^*(t;t_o,x_o))-W(t,x^*(t-;t_o,x_o))|$$

$$=\lim_{h\to 0+} |W(t,x^*(t;t_o,x_o))-W(t-h,x^*(t-h;t_o,xo))|,$$

since $W(t,x)$ is continuous and $x^*(.)$ is right continuous

$$=e^{-L\int_o^t \omega(s)ds} |V(t,x^*(t;t_o,x_o))-V(t, x^*(t-;t_o,x_o))|$$

$$\leq e^{-L\int_o^t \omega(s)ds} L|x^*(t;t_o,x_o) - x^*(t-;t_o,x_o)|$$

$$\leq e^{-L\int_o^t \omega(s)ds} L|G(t,x^*(t;t_o,x_o))| \; |u(t)-u(t-)|,$$

by (4.3.8)

$$\leq e^{-L\int_o^t \omega(s)ds} L \; g(t) \; |x^*(t;t_o,x_o)| \; |u(t)-u(t-)|$$

$$\leq e^{-L\int_o^t \omega(s)ds} L\rho g(t) \; |u(t)-u(t-)| \qquad (4.3.9)$$

Let $h_1 \geq h_2 \geq h_3 \geq \ldots > 0$ and let $A_k = [t-h_k, t]$ and $h_k \to 0$ as $k \to \infty$. Then $A_1 \supset A_2 \supset A_3 \supset \ldots$ and $\bigcap_{k=1}^{\infty} A_k = A_o$ where $A_o = \{t\}$. Therefore, $\mu(A_k) \to \mu(A_o)$ (Rudin [33], Th.1.19(e)). But $\mu(A_o) = |G(t,x^*(t;t_o,x_o))(u(t)-u(t-))|$, by Munroe [23, Ex.s,p.199]. Therefore (4.3.7) is established; and we have, from (4.3.6),

$$
\begin{aligned}
&|x^*(t;t_o,x_o)-x^*(t-;t_o,x_o)| \\
&=|G(t,x^*(t;t_o,x_o))(u(t)-u(t-))|
\end{aligned}
\tag{4.3.8}
$$

Now,

$$
\begin{aligned}
&|W(t,x^*(t;t_o,x_o))-W(t,x^*(t-;t_o,x_o))| \\
&=\lim_{h\to 0+} |W(t,x^*(t;t_o,x_o))-W(t-h,x^*(t-h;t_o,xo))|, \\
&\qquad \text{since } W(t,x) \text{ is continuous and } x^*(.) \text{ is} \\
&\qquad \text{right continuous} \\
&= e^{-L\int_o^t \omega(s)ds} |V(t,x^*(t;t_o,x_o))-V(t, x^*(t-;t_o,x_o))| \\
&\leq e^{-L\int_o^t \omega(s)ds} L|x^*(t;t_o,x_o) - x^*(t-;t_o,x_o)| \\
&\leq e^{-L\int_o^t \omega(s)ds} L|G(t,x^*(t;t_o,x_o))| \; |u(t)-u(t-)|, \\
&\qquad \text{by (4.3.8)} \\
&\leq e^{-L\int_o^t \omega(s)ds} L\; g(t) \; |x^*(t;t_o,x_o)| \; |u(t)-u(t-)| \\
&\leq e^{-L\int_o^t \omega(s)ds} L\rho g(t) \; |u(t)-u(t-)|
\end{aligned}
\tag{4.3.9}
$$

Therefore,

$$W(t_k, x^*(t_k; t_o, x_o))$$

$$\leq W(t_k, x^*(t_k-; t_o, x_o)) + e^{-L\int_o^{t_k}\omega(s)ds} L\rho g(t_k)|u(t_k) - u(t_k-)|$$

$$\leq W(t_{k-1}, x^*(t_{k-1}; t_o, x_o))\ e^{-\beta c(t_k - t_{k-1})} + e^{-L\int_o^{t_k}\omega(s)ds} \gamma L\rho g(t_k)\ e^{-c(t_k - t_o)},$$

by (4.3.5) and condition(4) of the theorem.

$$\leq e^{-\beta c(t_k - t_{k-1})} W(t_{k-1}, x^*(t_{k-1}; t_o, x_o)) + \gamma L\rho g(t_k)\, e^{-\beta c(t_k - t_o)}$$

since $0 < \beta < 1$. (4.3.10)

Now, from (4.3.5),

$$W(t, x^*(t; t_o, x_o))$$

$$\leq W(t_o, x_o)\ e^{-\beta c(t - t_o)} \quad \text{for } t \in [t_o, t_1) \tag{4.3.11}$$

and

$$W(t, x^*(t; t_o, x_o))$$

$$\leq W(t_1, x^*(t_1; t_o, x_o))\ e^{-\beta c(t - t_1)} \text{ for } t \in [t_1, t_2)$$

$$\leq [W(t_o, x_o) + \gamma L\rho g(t_1)]\ e^{-\beta c(t - t_o)} \text{ for } t \in [t_o, t_2),$$

by (4.3.10) and (4.3.11).

In general,

$$W(t, x^*(t; t_o, x_o))$$

$$\leq [W(t_o, x_o) + L\rho\gamma \sum_{k=1}^{\infty} g(t_k)]\ e^{-\beta c(t - t_o)}.$$

This implies

$$|x^*(t;t_o,x_o)|\,e^{-L\int_o^t \omega(s)ds} \le \left[K\,|x_o|\,e^{-L\int_o^{t_o}\omega(s)ds} + L\rho\alpha\sum_{k=1}^{\infty} g(t_k)\right] e^{-\beta c(t-t_o)};$$

therefore,

$$|x^*(t;t_o,x_o)| \le \left[K|x_o| + L\,\rho\alpha\sum_{k=1}^{\infty} g(t_k)\right] e^{\int_o^{\infty}\omega(s)ds}\; e^{-\beta c(t-t_o)} \tag{4.3.12}$$

Let δ_o ($\le \rho$) be such that if $|x_o| \le \delta_o$ then

$$K\,|x_o|\,e^{\int_o^{\infty}\omega(s)ds} < \delta_1, \text{ where } 0 < \delta_1 < \rho,$$

and let α be such that

$$L\,\rho\,\alpha\sum_{k=1}^{\infty} g(t_k)\quad e^{\int_o^{\infty}\omega(s)ds} \le \rho - \delta_1 .$$

The theorem then follows from (4.3.12).

CHAPTER 5

AN OPTIMAL CONTROL PROBLEM WITH VECTOR - VALUED COST FUNCTIONAL

5.1 INTRODUCTION

One of the important problems in optimal processes is the proper choice of the cost functional. In many cases it is a compromise worked out on several desirable characteristics. It seems quite reasonable that a compromise worked out on the basis of already obtained values of the characteristics is superior to a compromise of the cost functionals a priori. For example, consider a hierarchical optimal problem with two, say, cost functionals $J_1(u)$ and $J_2(u)$ where if the control minimizing cost functional $J_1(u)$ is not unique, one chooses a control which minimizes $J_2(u)$ among the controls which minimize $J_1(u)$. Let u^* be optimal control obtained in this manner. A control u_0 for which $J_1(u_0)$ is slightly greater $J_1(u^*)$ but $J_2(u_0)$ is considerably less than $J_2(u^*)$, is preferable to u^* in many cases. Keeping this consideration in view, an optimal control problem with vector-valued cost functional is formulated in this chapter.

5.2 FORMULATION OF OPTIMAL CONTROL PROBLEM

Let $[t_o,\beta]$ be a fixed interval, S a domain of R^n and Q a nonempty compact subset of R^m. Define

$$B = \bigcup_{t\in[t_o,\beta]} BV([\tau,t],S) , \tag{5.2.1}$$

and

$$U = \{ u\in BV([t_o,\beta],Q) \mid u \text{ right continuous} \} \tag{5.2.2}$$

Consider a control process governed by the measure differential equation

$$Dx = f(t,x_t,u(t)) + G(t,x_t,u(t))Du \tag{5.2.3}$$

where x_t is defined by (2.2.4), f is a functional defined for $t\in[t_o,\beta]$, $x \in B$, $u \in U$ with range in R^n and G is a functional defined for $t \in [t_o,\beta]$, $x \in B$, $u \in U$ with range in the space $\mathcal{M}$ of all n x m matrices . Let f and G satisfy the following assumptions :

A_o : For each $x \in B$, $u \in U$, $f(t,x_t, u(t))$ is Lebesgue measurable function of t and $G(t,x_t,u(t))$ is integrable function of t with respect to LS-measure du(t) ;

A_1 : $\lim_{k\to\infty} x^{(k)}(s) = x^*(s)$ for each $s \in [\tau,T]$

and

$\lim_{k\to\infty} u^{(k)}(s) = u^*(s)$ for each $s \in [t_o,T]$

imply

$$\lim_{k\to\infty} f(t,x_t^{(k)}, u^{(k)}(t)) = f(t,x_t^*,u^*(t))$$

for each $t \in [t_o,T]$

$(t_o < T \leq \beta)$

A_2 : There exists a real function r, defined on $[t_o,\beta]$ and integrable with respect to the Lebesgue measure, such that

$$|f(t,x_t,u(t))| \leq r(t)$$

uniformly with respect to $x \in B$, $u \in U$;

A_3 : If

$$\lim_{k\to\infty} t_k = t^*, \qquad t_k \in [t_o,T] ,$$

$$\lim_{k\to\infty} x^{(k)}(t) = x^*(t) \text{ for each } t \in [\alpha,T]$$

and

$$\lim_{k\to\infty} u^{(k)}(t) = u^*(t) \text{ for each } t \in [t_o,T]$$

then

$$\lim_{k\to\infty} G(t_k, x^{(k)}_{t_k}, u^{(k)}(t_k)) = G(t^*,x^*_{t^*} ,u^*(t^*))$$

$$(t_o < T \leq \beta)$$

A_4 : There exists a constant K such that

$$|G(t,x_t,u(t))| \leq K$$

uniformly with respect to $x \in B$, $u \in U$.

Let $\varphi \in BV([\alpha,t_o],S)$ be given. Under the assumptions A_o through A_4, (5.2.3) has a bounded variation solution x, for each choice of $u \in U$, satisfying the initial condition $x(t) = \varphi(t)$ for $t \in [\alpha,t_o]$. This follows from Theorem 3. If $u \in U$ defined on the interval $[t_o,t_1]$ is such that the corresponding solution $x = x(.;t_o,\varphi)$ of (5.2.3) is also defined on $[t_o,t_1]$

then u is called an admissible control and x is called the corresponding trajectory, and we define

$$J^{\nu}(u) = \int_{t_o}^{t_1} h^{\nu}(t,x(t),u(t))dt, \qquad \nu = 1,2,\ldots,\ell \tag{5.2.4}$$

where h^{ν} are continuous real functions defined on $[t_o,\beta] \times S \times Q$. The vector function J with components $J^1, J^2, \ldots, J^{\ell}$ is called cost functional of the system (5.2.3) and the vector $J(u) \in R^{\ell}$ is called the cost of the (admissible) control u. We introduce a partial ordering in the set $\{J(u) : u \text{ admissible}\}$ as follows :

DEFINITION :

$$J(u^*) \leq J(u), \quad u,u^* \text{ admissible}$$

if

$$J^{\nu}(u^*) \leq J^{\nu}(u) \text{ for each } \nu = 1,2,\ldots,\ell .$$

A target set is a family $\mathcal{J}$ of nonempty compact sets $T_t \subset R^n$ defined for $t \in [t_o,\beta]$. We say that an admissible control u defined on $[t_o,t_1]$ transfers the function $\varphi \in BV([\alpha,t_o],S)$ to the target set $\mathcal{J}$, if the trajectory $x = x(.;t_o,\varphi,u)$, corresponding to the initial function φ and the control u, satisfies the relation $x(t_1) \in T_{t_1}$. Let $\mathcal{A}$ be the set of all admissible controls which transfer φ to $\mathcal{J}$.

DEFINITION : A control $u^* \in \mathcal{A}$ will be called an optimal control if

$$J(u) \leq J(u^*) \text{ implies } J(u) = J(u^*)$$

for each $u \in \mathcal{A}$.

Thus, u^* is optimal if no cost is less than $J(u^*)$ i.e. if $J(u^*)$ is minimal (Halmos [16]) with respect to the partial ordering defined above.

The optimization problem consists in finding such an optimal control.

5.3 EXISTENCE OF AN OPTIMAL CONTROL

Theorem 10 :

Let a control problem be given with the following data :

(a) $Dx = f(t,x_t,u(t)) + G(t,x_t,u(t))Du$, $t \in [t_o,\beta]$ with f and G satisfying the assumptions A_o, A_1, A_2, A_3 and A_4 (of section 5.2)

(the differential constraint);

(b) the given initial function φ is an element of $BV([\gamma,t_o],S)$;

(c) $\mathcal{J}$, the target set, is a family of nonempty compact sets $T_t \subset R^n$ defined on $[t_o,\beta]$ and upper semicontinuous with respect to inclusion;

(d) $\mathcal{A}$ is the set of admissible controls u defined on subintervals $[t_o,t_1]$ contained in $[t_o,\beta]$ with the same left end point t_o (and perhaps different right end point $t_1 > t_o$) which transfer φ to $\mathcal{J}$ and further for each $u \in \mathcal{A}$

$$|\Delta u| \leq \Delta w$$

on each subinterval of $[t_o,t_1]$ where w is a

given nondecreasing right continuous real function defined on $[t_o,\beta]$ (the symbol Δw on the interval $[a,b]$, say, denotes $w(b)-w(a)$)

(e) $J = (J^1,\ldots,J^\ell)$ is the cost functional where J^ν are defined by

$$J^\nu(u) = \int_{t_o}^{t} h^\nu(t,x(t),u(t))dt, \nu=1,2,..\ell$$

h^ν being real continuous functions defined on $[t_o,\beta] \times S \times Q$;

(f) $\mathcal{A}$ is nonempty.

The control problem satisfying the conditions given above has an optimal control in $\mathcal{A}$.

PROOF : Since for each $u \in \mathcal{A}$, $|\Delta u| \leq \Delta w$ on each subinterval of $[t_o,t_1]$, therefore $u \in \mathcal{A}$ are of uniform bounded variation. We shall show that the corresponding trajectories x are also of uniform bounded variation. We have

$$x(t) = \begin{cases} \varphi(t) \text{ for } t \in [\tau,t_o] \\ \varphi(t_o)+\int_{t_o}^{t} f(s,x_s,u(s))ds \\ \quad +\int_{t_o}^{t} G(s,x_s,u(s))du(s) \text{ for } t \in (t_o,t_1] \end{cases} \tag{5.3.1}$$

Therefore,

$$\begin{aligned} v(x,[t_o,t_1]) &\leq \int_{t_o}^{t_1} |f(s,x_s,u(s))|ds \\ &\quad +\int_{t_o}^{t_1} |G(s,x_s,u(s))|dv_u(s) \\ &\leq \int_{t_o}^{t_1} r(s)ds + Kv(u,[t_o,t_1]), \end{aligned}$$

by A_2 and A_4

From this we obtain

$$v(x, [t_o,t_1]) \leq \int_{t_o}^{\beta} r(s)ds + K(w(\beta)-w(t_o)) \quad (5.3.2)$$

which shows that $v(x,[t_o,t_1])$ are uniformly bounded for all trajectories x corresponding to controls in $\mathcal{A}$. Hence the trajectories x are also uniformly bounded.

Now, since $\mathcal{A}$ is nonempty, and the corresponding trajectories are uniformly bounded, it follows that for each $\nu = 1, 2, \ldots\ldots, \ell$,

$$\inf_{u \in \mathcal{A}} J^{\nu}(u) = \widetilde{m}^{\nu} \quad (5.3.3)$$

is finite .

Since $\mathcal{A}$ is nonempty, let $u_o \in \mathcal{A}$. Let

$$P = \{ J(u) \mid J(u) \leq J(u_o),\ u \in \mathcal{A} \} \quad (5.3.4)$$

so that $P \subset J(\mathcal{A}) \subset R^{\ell}$. P is not empty since $J(u_o) \in P$. Our aim now is to show that every chain (Halmos [16]) in P has infimum (greatest lower bound) belonging to P and then to apply Zorn's lemma to show that P contains a minimal element $J(u^{**})$, say. u^{**} will then be an optimal control.

Let $\mathcal{C}$ be any chain in P. If $\mathcal{C}$ is finite it obviously contains the infimum. Let us consider the case when $\mathcal{C}$ is infinite. We have, from (5.3.3), for each $\nu = 1,2,\ldots,\ell$, the set $\{ J^{\nu}(u) \mid u \in \mathcal{C} \}$ is bounded below by $\widetilde{m}^{\nu}$ and so its infimum exists. Let

We can select from $\mathcal{A}$ a sequence of controls

$$u^{(k)} \text{ on } [t_o, t_1^{(k)}] \text{ with } J(u^{(k)}) \in \mathcal{C} \qquad (5.3.6)$$

such that $J^{\nu}(u^{(k)})$ decreases monotonically to c^{ν} ($\nu = 1,2,\ldots, \ell$). To see this, let (5.3.6) be so selected that $J^1(u^{(k)})$ decreases monotonically to c^1, and suppose for some ν_0, $2 \leq \nu_0 \leq \ell$,

$$J^{\nu_0}(u^{(k)}) \not\to c^{\nu_0}$$

and

$$J^{\nu}(u^{(k)}) \to c^{\nu} \text{ for } \nu = 1,2,\ldots,\nu_0 - 1 ;$$

then select another sequence $u_*^{(k)}$ such that

$$J^{\nu_0}(u_*^{(k)}) \to c^{\nu_0} .$$

Then we shall also have, for this sequence,

$$J^{\nu}(u_*^{(k)}) \to c^{\nu} \text{ for } \nu = 1,2,\ldots,\nu_0 - 1 ;$$

for, we have $J^{\nu_0}(u_*^{(k)}) \leq J^{\nu_0}(u^{(k)})$ for $k \geq k_o$ for some k_o and, therefore, since $\mathcal{C}$ is a <u>chain</u>,

$$J^{\nu}(u_*^{(k)}) \leq J^{\nu}(u^{(k)}), \quad \nu = 1,2,\ldots,\nu_0 - 1$$

so that $\lim\limits_{k \to \infty} J^{\nu}(u_*^{(k)}) = c^{\nu}$, $\nu = 1,2,\ldots,\nu_0 - 1$.

Proceeding this way the sequence (5.3.6) can finally be selected.

Now we select a subsequence of (5.3.6), still to be called $u^{(k)}$, such that $t_1^{(k)} \to t_1^*$ monotonically. Consider the case when $\{t_1^{(k)}\}$ is monotonic decreasing (the case when it is increasing will be considered later). Next choose $\hat{t}_1$ such that $\hat{t}_1 = t_1$ if $t_1^{(k)} = t_1$ for all k, otherwise let $t_1^* < t_1^{(k_o+1)} \leq \hat{t}_1 < t_1^{(k_o)} < \beta$ for some k_o.

(From now on all references to the index k tacitly assumes $k > k_o$). We define the extended control $\hat{u}^{(k)}$ on $[t_o, \hat{t}_1]$ by

$$\hat{u}^{(k)}(t) = \begin{cases} u^{(k)}(t) & \text{if } t \in [t_o, t_1^{(k)}] \\ u^{(k)}(t_1^{(k)}) & \text{if } t \in [(t_1^{(k)}, \hat{t}_1] \end{cases} \tag{5.3.7}$$

Since $u^{(k)}(t) \in Q$ for all $t \in [t_o, t_1^{(k)}]$, $\hat{u}^{(k)}(t)$ also $\in Q$ for $t \in [t_o, \hat{t}_1]$. Evidently $\hat{u}^{(k)}$ are right continuous on $[t_o, \hat{t}_1)$, and $|\Delta \hat{u}^{(k)}| \leq \Delta w$ on every subinterval of $[t_o, \hat{t}_1]$. Now, by Helly's selection principle, there exists a subsequence (for which we shall use the same notation $\hat{u}^{(k)}$) and a function of bounded variation u^* such that

$$\lim_{k \to \infty} \hat{u}^{(k)}(t) = u^*(t) \tag{5.3.8}$$

everywhere on $[t_o, \hat{t}_1]$. It will be proved that $u^* \in \mathcal{A}$. This will be done in several steps.

Step I : We shall first show that u^* is right continuous. Let τ be any point in $[t_o, \hat{t}_1)$. Since $|\Delta \hat{u}^{(k)}| \leq \Delta w$ on every subinterval of $[t_o, \hat{t}_1]$, we have

$$|\hat{u}^{(k)}(\tau + s) - \hat{u}^{(k)}(\tau)| \leq w(\tau + s) - w(\tau), \quad \tau \leq \tau + s \leq \hat{t}_1,$$

and since w is right continuous at τ, it follows that given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any k,

$$|\hat{u}^{(k)}(\tau + s) - \hat{u}^{(k)}(\tau)| < \varepsilon \text{ for } 0 \leq s \leq \delta .$$

Therefore,

$$\lim_{s\to 0+} \hat{u}^{(k)}(\tau+s) = \hat{u}^{(k)}(\tau) \tag{5.3.9}$$

uniformly in k. Also,

$$\lim_{k\to\infty} \hat{u}^{(k)}(\tau+s) = u^*(\tau+s) .$$

Hence, applying the Moore theorem on interchange of order of repeated limits, we obtain

$$\lim_{s\to 0+} \lim_{k\to\infty} \hat{u}^{(k)}(\tau+s) = \lim_{k\to\infty} \lim_{s\to 0+} \hat{u}^{(k)}(\tau+s)$$

i.e., $\quad u^*(\tau+0) = u^*(\tau) .$

Thus, u^* is right continuous at τ. But τ was taken arbitrarily in $[t_o,\hat{t}_1)$. Hence u^* is right continuous at each $t \in [t_o,\hat{t}_1)$.

Step II : Since $\hat{u}^{(k)}(t) \in Q$ for $t \in [t_o,\hat{t}_1]$ and Q is compact, it follows from (5.3.8) that $u^*(t)$ also $\in Q$ for each $t \in [t_o,\hat{t}_1]$. Since $\Delta u^* = \lim\limits_{k\to\infty} \Delta \hat{u}^{(k)}$ on every subinterval of $[t_o,\hat{t}_1]$, it follows that given any $\varepsilon > 0$,

$$\Delta u^* \leq |\Delta \hat{u}^{(k)}| + \varepsilon \quad \text{for sufficiently large k}$$

$$\leq \Delta w + \varepsilon .$$

Since ε is arbitrary, we get

$$\Delta u^* \leq \Delta w$$

on every subinterval of $[t_o,\hat{t}_1]$.

Step III : Let $x^{(k)}$ be the trajectory defined on $[t_o, t_1^{(k)}]$ corresponding to the control $u^{(k)}$, $(k=1,2,\ldots)$. The extended control $\hat{u}^{(k)}$ coincides with $u^{(k)}$ on $[t_o, t_1^{(k)}]$ and is constant and, therefore, continuous on $[t_1^{(k)}, \hat{t}_1]$ (see (5.3.7)). Hence, by Theorem 4, $x^{(k)}$ can be extended to $[t_o, \hat{t}_1]$. The extended trajectory $\hat{x}^{(k)}$ is given by

$$\hat{x}^{(k)}(t) = \begin{cases} \varphi(t) \text{ for } t \in [\tau, t_o] \\ \varphi(t_o) + \int_{t_o}^{t} f(s, \hat{x}_s^{(k)}, \hat{u}^{(k)}(s))ds \\ \quad + \int_{t_o}^{t} G(s, \hat{x}_s^{(k)}, \hat{u}^{(k)}(s))d\hat{u}^{(k)}(s) \\ \qquad \text{for } t \in (t_o, \hat{t}_1] \end{cases} \tag{5.3.11}$$

Since

$$v(\hat{u}^{(k)}, [t_o, \hat{t}_1]) = v(u^{(k)}, [t_o, t_1^{(k)}]),$$

total variations of $\hat{x}^{(k)}$ on $[t_o, \hat{t}_1]$ can be seen, as in (5.3.2), to be uniformly bounded. $\hat{x}^{(k)}$ are also uniformly bounded on $[t_o, \hat{t}_1]$. Hence there exists a subsequence (still to be labelled $\hat{x}^{(k)}$) and a function x^* such that

$$\lim_{k\to\infty} \hat{x}^{(k)}(t) = x^*(t) \tag{5.3.12}$$

everywhere on $[t_o, \hat{t}_1]$. By selecting the corresponding subsequence from $\hat{u}^{(k)}$ we do not change any of the preceding limiting operations satisfied by $\hat{u}^{(k)}$. From (5.3.8), (5.3.12), A_1 and A_2, we have, by Lebesgue's dominated convergence theorem,

$$\lim_{k\to\infty} \int_{t_o}^{t} f(s, \hat{x}_s^{(k)}, \hat{u}^{(k)}(s))ds = \int_{t_o}^{t} f(s, x_s^*, u^*(s))\, ds \tag{5.3.13}$$

for all $t \in [t_o, \hat{t}_1]$.

Now, $|\Delta \hat{u}^{(k)}| \leq \Delta w$, $|\Delta u^*| \leq \Delta w$ on every subinterval of $[t_o, \hat{t}_1]$; and $\hat{u}^{(k)}(t) \to u^*(t)$ everywhere on $[t_o, \hat{t}_1]$. Furthermore,

$$\left| \int_{t_1}^{t_2} G_j^i (t, \hat{x}_t^{(k)}, \hat{u}^{(k)}(t)) dw(t) \right|$$

$$\leq \int_{t_1}^{t_2} \left| G_j^i (t, \hat{x}_t^{(k)}, \hat{u}^{(k)}(t)) \right| dw(t)$$

$$\leq \int_{t_1}^{t_2} \left| G(t, \hat{x}_t^{(k)}, \hat{u}^{(k)}(t) \right| dw(t)$$

$$\leq \int_{t_1}^{t_2} K dw(t), \text{ by } A_4$$

$$= K (w(t_2) - w(t_1)),$$

and therefore the integrals $\int_{t_o}^{t} G_j^i (s, \hat{x}_s^{(k)}, \hat{u}^{(k)}(s)) dw(s)$ are absolutely continuous with respect to $dw(s)$ uniformly in k, and bounded uniformly. Also, by A_3,

$$\lim_{k \to \infty} G(s, \hat{x}_s^{(k)}, \hat{u}^{(k)}(s)) = G(s, x_s^*, u^*(s)).$$

Hence, by (1.2.11), we have

$$\lim_{k \to \infty} \int_{t_o}^{t} G(s, \hat{x}_s^{(k)}, \hat{u}^{(k)}(s)) d\hat{u}^{(k)}(s)$$

$$= \int_{t_o}^{t} G(s, x_s^*, u^*(s)) du^*(s) \qquad (5.3.14)$$

for all $t \in [t_o, \hat{t}_1]$. By (5.3.11), (5.3.12), (5.3.13) and (5.3.14), we obtain

$$x^*(t) = \begin{cases} \varphi(t) \text{ for } t \in [\alpha, t_o] \\ \varphi(t_o) + \int_{t_o}^{t} f(s, x_s^*, u^*(s))\, ds \\ \quad + \int_{t_o}^{t} G(s, x_s^*, u^*(s))\, du^*(s) \\ \qquad \text{for } t \in (t_o, \hat{t}_1] \end{cases} \tag{5.3.15}$$

i.e. x^* is the solution of (5.2.3) with initial function φ corresponding to the control u^*.

Step IV : We shall now show that

$$\lim_{k \to \infty} \hat{x}^{(k)}(t_1^{(k)}) = x^*(t_1^*) \tag{5.3.16}$$

and that $x^*(t_1^*) \in T_{t_1^*}$.

We have,

$$| x^{(k)}(t_1^{(k)}) - x^*(t_1^*) |$$

$$\leq | x^{(k)}(t_1^{(k)}) - x^*(t_1^{(k)}) |$$

$$+ | x^*(t_1^{(k)}) - x^*(t_1^*) | \tag{5.3.17}$$

Since u^* is right continuous on $[t_o, \hat{t}_1]$, x^* given by (5.3.15) is also right continuous on $[t_o, \hat{t}_1]$ .Therefore, $x^*(t_1^{(k)}) \to x^*(t_1^*)$ and hence the last term on the right in (5.3.17) approaches zero as $k \to \infty$.

Further,

$$x^{(k)}(t_1^{(k)}) - x^* (t_1^{(k)})$$

$$= \int_{t_o}^{t_1^{(k)}} f(s,x_s^{(k)}, u^{(k)}(s))ds$$

$$+ \int_{t_o}^{t_1^{(k)}} G(s,x_s^{(k)}, u^{(k)} (s))du^{(k)}(s)$$

$$- \int_{t_o}^{t_1^{(k)}} f(s,x_s^*, u^*(s))ds-\int_{t_o}^{t_1^{(k)}} G(s,x_s^*,u^*(s))du^*(s)$$

$$= \int_{t_o}^{t_1^*} \left[f(s,x_s^{(k)}, u^{(k)}(s)) - f(s,x_s^*,u^*(s))\right] ds$$

$$+ \left[\int_{t_o}^{t_1^*} G(s,x_s^{(k)},u^{(k)}(s))du^{(k)}(s)\right.$$

$$\left. - \int_{t_o}^{t_1^*} G(s,x_s^*, u^*(s))du^*(s) \right]$$

$$+ \int_{t_1^*}^{t_1^{(k)}} \left[f(s,x_s^{(k)}, u^{(k)}(s)) - f(s,x_s^*,u^*(s))\right] ds$$

$$+ \left[\int_{t_1^*}^{t_1^{(k)}} G(s,x_s^{(k)},u^{(k)}(s)) du^{(k)} (s)\right.$$

$$\left. - \int_{t_1^*}^{t_1^{(k)}} G(s,x_s^*,u^*(s)) du^*(s) \right]$$

$$= I_1 + I_2 + I_3 + I_4, \text{ say.}$$

By (5.3.13) and (5.3.14) , we have

$$I_1 \to 0, \ I_2 \to 0 \text{ as } k \to \infty .$$

We have, further,

$$| I_3 | \leq \int_{t_1^*}^{t_1^{(k)}} | f(s, x_s^{(k)}, u^{(k)}(s)) | ds$$

$$+ \int_{t_1^*}^{t_1^{(k)}} |f(s, x_s^*, u^*(s))| \; ds$$

$$\leq \; 2 \int_{t_1^*}^{t_1^{(k)}} r(s) ds \to 0 \text{ as } k \to \infty \, ;$$

$$| I_4 | \; \leq \int_{t_1^*}^{t_1^{(k)}} |G(s, x_s^{(k)}, u^{(k)}(s))| dw(s)$$

$$+ \int_{t_1^*}^{t_1^{(k)}} |G(s, x_s^*, u^*(s))| dw(s)$$

$$\leq 2 \, K \left[w(t_1^{(k)}) - w(t_1^*) \right] \to 0 \text{ as } k \to \infty \, ,$$

since w is right continuous. Thus (5.3.16) is established.

$$\text{Now}, \; x^{(k)} (t_1^{(k)}) \in T_{t_1^{(k)}} \text{ for } k = 1, 2, \ldots,$$

and $x^* (t_1^*) = \lim_{k\to\infty} x^{(k)} (t_1^{(k)})$. If $x^*(t_1^*)$ were not in $T_{t_1^*}$ then there would exist a neighbourhood N of the compact set $T_{t_1^*}$ so that $x^*(t_1^*)$ is not in the closure $\overline{N}$ of N. But since T_t is upper semicontinuous, $T_t \subset N$ for t sufficiently near t_1^*. Thus $x^{(k)} (t_1^{(k)}) \in N$ for large k and yet $x^*(t_1^*) \notin \overline{N}$. This contradiction shows that $x^*(t_1^*) \in T_{t^*}$.

Summing up the results in Steps I, II, III and IV, it follows that u^* on $[t_o, t_1^*]$ belongs to $\mathcal{A}$. It will now be proved that $J(u^*)$ is infimum of $\mathcal{C}$. We have, for each $\nu = 1,2,\dots,\ell$,

$$J^\nu(u^{(k)}) = \int_{t_o}^{t_1^{(k)}} h^\nu\,(s,\, x^{(k)}(s),\, u^{(k)}(s))ds$$

$$= \int_{t_o}^{t_1^*} h^\nu\,(s,x^{(k)}(s),u^{(k)}(s))ds + \int_{t_1^*}^{t_1^{(k)}} h^\nu\,(s,x^{(k)}(s),\, u^{(k)}(s))ds \qquad (5.3.18)$$

The continuity of h^ν and uniform boundedness of $x^{(k)}$ and $u^{(k)}$ imply that $h^\nu\,(s,\, x^{(k)}(s),\, u^{(k)}\,(s))$ is also uniformly bounded on $[t_o,\, t_1^{(k)}]$ and so the last integral in (5.3.18) approaches zero, and since

$$\lim_{k\to\infty} x^{(k)}\,(t) = x^*(t),\ \lim_{k\to\infty} u^{(k)}\,(t) = u^*(t)$$

everywhere on $[t_o,t_1^*]$, therefore, by applying Lebesgue's dominated convergence theorem to the first integral in (5.3.18), we obtain

$$\lim_{k\to\infty} J^\nu(u^{(k)}) = \int_{t_o}^{t_1^*} h^\nu\,(s,x^*(s),u^*(s))ds,$$

$$= J^\nu\,(u^*) \qquad (5.3.19)$$

$\nu = 1,2,\dots,\ell.$

By (5.3.5) and (5.3.19), we get

$$J^\nu(u^*) = c^\nu \quad , \nu = 1,2,\ \dots,\ \ell.$$

Thus $J(u^*) \in P$ is infimum of $\mathcal{C}$. Hence, by Zorn's lemma, P will contain a minimal element, say $J(u^{**})$. This proves the existence of an optimal control u^{**} in $\mathcal{A}$.

We shall now consider the case when $\{t_1^{(k)}\}$ is monotonic increasing. We extend all the controls $u^{(k)}$ to the interval $[t_o, \hat{t}_1]$, where $\hat{t}_1 = t_1^* + \delta$ for appropriately small $\delta > 0$, by defining

$$\hat{u}^{(k)}(t) = \begin{cases} u^{(k)}(t) \text{ for } t \in [t_o, t_1^{(k)}] \\ u^{(k)}(t_1^{(k)}) \text{ for } t \in (t_1^{(k)}, \hat{t}_1] \end{cases} \quad (5.3.20)$$

As before, there exists a subsequence $\hat{u}^{(k)}$ (without changing the notation) and a right continuous function u^* of bounded variation such that everywhere on $[t_o, \hat{t}_1]$,

$$\lim_{k\to\infty} \hat{u}^{(k)}(t) = u^*(t) \quad (5.3.21)$$

and

$$\lim_{k\to\infty} \hat{x}^{(k)}(t) = x^*(t) \quad (5.3.22)$$

where $\hat{x}^{(k)}$ and x^* are the trajectories corresponding to $\hat{u}^{(k)}$ and u^* respectively.

We shall show that

$$\lim_{k\to\infty} x^{(k)}(t_1^{(k)}) = x^*(t_1^*). \quad (5.3.23)$$

Consider

$$|x^{(k)}(t_1^{(k)}) - x^*(t_1^*)|$$

$$\leq |x^{(k)}(t_1^{(k)}) - \hat{x}^{(k)}(t_1^*)|$$

$$+ |\hat{x}^{(k)}(t_1^*) - x^*(t_1^*)|. \quad (5.3.24)$$

The last term on the right of (5.3.24) approaches zero as $k \to \infty$ because of (5.3.22). For the first term on the right, we write

$$
\begin{aligned}
& x^{(k)}(t_1^{(k)}) - \hat{x}^{(k)}(t_1^*) \\
&= \int_{t_0}^{t_1^{(k)}} f(s, x_s^{(k)}, \hat{u}^{(k)}(s))ds \\
&\quad + \int_{t_0}^{t_1^{(k)}} G(s, x_s^{(k)}, \hat{u}^{(k)}(s))\, d\hat{u}^{(k)}(s) \\
&\quad - \int_{t_0}^{t_1^*} f(s, \hat{x}_s^{(k)}, \hat{u}^{(k)}(s))ds \\
&\quad - \int_{t_0}^{t_1^*} G(s, \hat{x}_s^{(k)}, \hat{u}^{(k)}(s))d\hat{u}^{(k)}(s) \\
&= - \int_{t_1^{(k)}}^{t_1^*} f(s, \hat{x}_s^{(k)}, \hat{u}^{(k)}(s))ds \\
&\quad - \int_{t_1^{(k)}}^{t_1^*} G(s, \hat{x}_s^{(k)}, \hat{u}^{(k)}(s))\, d\hat{u}^{(k)}(s) \\
&= I_1 + I_2 \text{, say.}
\end{aligned}
$$

Then

$$
|I_1| \leq \int_{t_1^{(k)}}^{t_1^*} |f(s, \hat{x}_s^{(k)}, \hat{u}^{(k)}(s))|ds \leq \int_{t_1^{(k)}}^{t_1^*} r(s)ds
$$

$$
\to 0 \text{ as } k \to \infty;
$$

and

$$
I_2 = - \int_{t_1^{(k)}}^{t_1^*} G(s, \hat{x}_s^{(k)}, \hat{u}^{(k)}(s))d\hat{u}^{(k)}(s) = 0
$$

for each k, since $\hat{u}^{(k)}$ is constant on $[t_1^{(k)}, t_1^*]$ (see (5.3.20)). Thus (5.3.23) is established. The rest of the argument proceeds just as before. This completes the proof.

BIBLIOGRAPHY

1. Barbashin, E.A. : On Stability with respect to Impulsive Perturbations, Differentsial'nye Uravneniya, Vol.2,No.7,(1966), pp. 863-871.

2. Bellman, R., and Cooke, K.L. : 'Differential-Difference Equations,' Academic Press, New York and London, 1963.

3. Choksy, N. H. : Time lag systems - a bibliography, I.R.E. Trans. Automat. Control, AC 5 (1960),pp. 66-70.

4. Chyung, D. H. : Optimal Systems with multiple Cost Functionals, J. SIAM Control, 5 (1967), pp. 345-351.

5. Coddington, E.A., and Levinson, N. : 'Theory of Ordinary Differential Equations', McGraw-Hill, New York, 1955.

6. Das, P.C., and Sharma, R.R. : On Optimal Controls for Measure Delay-Differential Equations, to appear in J.SIAM Control.

7. Dunford, N., and Schwartz, J.T. : 'Linear Operators, Part I : General Theory,' Interscience, New York, 1964.

8. El'sgol'ts, L.E. : 'Introduction to the Theory of Differential Equations with Deviating Arguments,' Holden-Day, San Francisco, 1966.

9. El'sgol'ts, L.E. : 'Qualitative Methods in Mathematical Analysis', Amer. Math. Soc., Providence, Rhode Island, 1964.

10. Filippov, A.F. : On Certain Questions in the Theory of Optimal Control, J. SIAM Control, 1 (1962), pp. 76-84.

11. Graves, L.M. : 'The Theory of Functions of Real Variables,' McGraw-Hill, New York, 1956.

12. Gromova, P.S. : Some questions on the qualitative theory of integro-differential equations with deviating argument, Trudy Sem. Teor. Differencial. Uravneni s Otklon. Argumentom Univ. Druzby Narodov Patrisa Lumumby 5 (1967), pp. 61-76.

13. Halanay, A. : 'Differential Equations : Stability, Oscillations, Time Lags', Academic Press, New York, 1965.

14. Halanay, A., and Wexler, D. : 'Theoria Calitativa a systemolor en impulsuri', Editura Academiei Republicii Socialiste Romania Bucharest, 1968.

15. Halmos, P.R. : 'Measure Theory', Van Nostrand, East-West Student Edition, 1964.

16. Halmos, P.R. : 'Naive Set Theory', Van Nostrand, Princeton, N.J., 1960.

17. Halperin, I. : 'Introduction to the Theory of Distributions, University of Toronto Press, Toronto, 1952.

18. Krasovskii, N.N. : 'Stability of Motion,' Stanford Univ. Press, Stanford, Calif., 1963.

19. Lakshmikantham, V., and Leela, S. : 'Differential and Integral Inequalities, Theory and Applications, Vol.1,' Academic Press, New York, 1969.

20. Lee, E.B., and Markus, L. : 'Foundations of Optimum Control Theory,' Wiley, New York, 1967.

21. Levin, J.J. : Boundedness and Oscillation of Some Volterra and Delay Equations, J.Diff.Eqns., 5 (1969), pp. 369 - 398.

22. McShane, E.J. : 'Integration', Princeton University Press, Princeton, N.J., 1944.

23. Munroe, M.E. : 'Introduction to Measure and Integration', Addison - Wesley, Reading, Massachusetts, 1953.

24. Myshkis, A.D. : 'Linear Differentialgleichungen mit nacheilondem Argument,' Deutscher Verlag der Wiss. Berlin, 1955.

25. Myshkis, A.D. : A. M. Samoilenko Systems with impulses at given instants of time, Math. Sb(N.S.) 74 (116) (1967), pp. 202 - 208.

26. Natanson, I.P. : 'Theory of functions of a Real Variable, Vol. 1', Frederick Ungar Publishing Co., New York, 1955.

27. Oguztöreli, M.N. : 'Time-Lag Control Systems,' 'Academic Press, New York and London, 1966.

28. Pavlidis, T. : Optimal Control of Pulse Frequency Modulated Systems, IEEE Trans. Automat. Control, AC - 11 (1966), pp. 676 - 684.

29. Pavlidis, T. : Stability of Systems described by Differential Equations Containing Impulses, IEEE Trans. Automat. Control, AC - 12 (1967), pp. 43 - 45.

30. Pinney, E. : 'Differential - Difference Equations,' University of California Press, Berkeley, 1958.

31. Pontryagin, L.S., Boltyanskii, V.G., Gamkrelidze, R.V., and Mishchenko, E.F. : 'The Mathematical Theory of Optimal Processes,' Interscience, New York, 1962.

32. Rishel, R.W. : An Extended Pontyagin Principle for Control Systems whose control laws contains measures, J. SIAM Control, 3 (1965), pp. 191 - 205.

33. Rudin, W. : 'Real and Complex Analysis,' Mc-Graw Hill, New York, 1966.

34. Schmaedeke, W.W. : Optimal Control Theory for Nonlinear Vector Differential Equations containing Measures, J. SIAM Control, 3 (1965), pp. 231 - 280.

35. Schmaedeke, W.W. : The Existence Theory of Optimal Control Systems, 'Advances in Control Systems', Vol.3, pp. 111-149, Academic Press, New York, 1966.

36. 'Studies in Integro - differential Equations in Kirghizia', No. 1 (1961), No. 2 (1962), No. 3 (1965), No.4 (1967), Izdat. Akad. Nauk Kirgiz. SSR, Frunze.

37. Volterra, V. : 'Theory of Functionals and of Integral and Integro - Differential Equations,' Dover, New York, 1959.

38. Yoshizawa, T. : Stability Theory By Liapunov's Second Method, Math. Soc. Japan, 1966.

39. Yosida, K. : 'Functional Analysis,' Springer-Verlag, Berlin, Heidelberg, New York, 1968.

40. Zabalishchin, S.T. : Stability of Generalized Processes, Differentisial'nye Uravneniya, Vol. 2, No. 7 (1966), pp. 872 - 881.

41. Zadeh, L.A. : Optimality and Non-scalar-valued Performance Criteria, IEEE Trans. Automat. Control, AC - 8 (1963), pp. 59 - 60.